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SPACE-TIME BOUNDARIES FOR RANDOM WALKS OBTAINED FROM DIFFUSE MEASURES

BY

DAVID HANDELMAN*

Mathematics Department
University of Ottawa, Ottawa, Ontario K1N 6N5, Canada
email:dehsg@ACADVM1.uottawa.ca

ABSTRACT

We define a "space-time" boundary (referring to space-time harmonic functions) to encompass random walks obtained from compactly supported diffuse measures on Euclidean space, and then prove that in many cases, a qualitative analogue of the Ney–Spitzer theorem (1966) holds, namely that the space-time boundary admits a natural identification with the convex hull of the support of the measure. This can also be interpreted as a generalization to the diffuse case of the weighted moment mapping of algebraic geometry. In many more cases, a weaker analogue holds, identifying the faithful extreme space-time harmonic functions with the interior of the convex body.

Introduction

A well known theme in the theory of random walks (on discrete abelian groups) is the study of their harmonic functions. The space of suitably normalized harmonic functions forms a convex set; however, this is only a slice of a much larger and more useful space consisting of the space-time harmonic functions. This itself is only a subset (often open and dense) of a natural compact set, in fact, a Choquet simplex, consisting of space-time harmonic functions defined on a suitable space-time cone determined by an initial distribution. The extremal boundary of this Choquet simplex is a very interesting object. For example, if the random walk is

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an iid with finite support on $\mathbf{Z}^d$, the extremal boundary is naturally homeomorphic to the convex hull of the support, and the natural map implementing this is essentially the weighted moment map ([Od])—paradoxically, this set of extreme points of a convex set can be identified with the set of all points of a convex polytope, and the resulting facial structure can be exploited.

Still in the case of an iid with finite support on $\mathbf{Z}^d$, the harmonic functions that are extremal can be identified with the boundary of a log-convex set—the set of nonnegative solutions to $P(x) = 1$ where $x = \exp X$ is in the positive orthant of Euclidean space, $(\mathbf{R}^d)^{++}$ and P is the "real slice" of the Fourier transform of the measure, viewed as a Laurent polynomial in x. The remaining extremal space-time harmonic functions are also eigenvectors (for the multiplication operator) and can be identified with the boundary of the log-convex sets $\{x \in (\mathbf{R}^d)^{++}$ such that $P(x) = \lambda\}$ where λ varies over the half-open real interval $[\lambda_0, \infty)$, with $\lambda_0 = \inf\{P(x)$ such that $x \in (\mathbf{R}^d)^{++}\}$. Finally, the set of extreme points of the whole space-time boundary (which will be defined below in detail) includes all these, together with their limit points. These turn out to be space-time harmonic functions defined on a proper space-time cone; moreover, there is a natural identification of this set as the boundary of the convex hull of the support of the original measure. In this context, there is a qualitative interpretation of the Ney–Spitzer theorem [NS]. For each $\lambda > \lambda_0$, consider the set of solutions to $P(x) = \lambda$, and normalize in some way, so the volumes they enclose are more or less comparable as $\lambda \to \infty$. Then after possible translation, these shells converge to the boundary of the convex hull of the support of the original measure. This could have been deduced from the Ney–Spitzer theorem (which involved estimates of the distributions of high convolution powers of the measure) had there not been an initial distortion which had the effect of making the boundary of the convex polytope into a sphere.

In any event, in the case of an iid μ with finite support on $\mathbf{Z}^d$ (with initial distribution a singleton, for example), there is a natural identification of the extremal space-time boundary with the convex hull of the support of the measure. If we examine the mapping which implements the identification, we also see that it is precisely the weighted moment map of algebraic geometry, [GS], [A], [Od; Remark, p. 94]. It also occurs in the study of an algebraic invariant for lattice polytopes, which in turn arose from attempts to understand an eventual positivity problem for products of polynomials, [H4] and [H8]; the latter class of problems was naturally suggested by problems in the computation of the ordered K_0 theory of actions of tori on certain C*-algebras. For extensions of this to compact Lie group actions, see [H1, H2, H5, H7, H9].

The portion of the space-time boundary corresponding to the interior of the convex body is described by a special case of the Legendre transform of convex analysis, [Ro; Theorem 26.5] (a version earlier than this but after [NS] is proved by Rothaus, [Rh]). In this class of examples, the Martin (exit) boundary consists of a very small cross-section, just the solutions to $P(x) = 1$ (and this is insensitive to the choice of initial distribution).

These results cited up to this point concern the discrete case, that is, an iid with finite support. One approach adopted in [H4] and [H8], and extending considerably an idea in [DSW], is to identify the space-time boundary with positive multiplicative functionals on an algebra naturally associated to the process. (From another point of view, this boundary is just the Choquet boundary on a dimension group attached to the random walk.)

If instead, μ is not supported on a lattice, e.g., if μ is the restriction of Lebesgue measure to a compact convex body, it is not clear how to define the space-time boundary so that the analogous results hold, at least part of the time. There are several possible choices. One definition involving $\mathbb{L}^\infty$ functions gives a space occurring as the set of positive multiplicative functions on an enormous partially ordered algebra. This algebra is much too large to deal with, and the strongest possible generalization of the main result from the discrete case (that there is a natural homeomorphism to the convex hull of the support) fails except in degenerate cases, which amount to the original discrete form. A much smaller and more appropriate candidate arises from the use of continuous functions; the problem is that the algebra (whose positive multiplicative functionals correspond to the points of the space-time boundary) need not exist! However, in many reasonable cases, existence can be established. It turns out that when this algebra exists, there is a natural map from the space-time boundary to the convex hull of the support, but in general, it is not a homeomorphism. Again, in reasonable cases (Lebesgue measure on a polytope or a planar convex set, for example), the map is a homeomorphism and the weighted moment mapping theorem generalizes exactly. The map also exists and is a homeomorphism if $K := \mathrm{cvx}\,\mathrm{supp}\,\mu$ is strictly convex (no line segments in the boundary) and μ is absolutely continuous with respect to Lebesgue measure.

If line segments are permitted in the boundary, then there are obstructions to the map being a homeomorphism, and moreover, this property is sensitive to slight deformations of K. An example is a one parameter family of measures in $\mathbf{R}^2$ whose convex supports resemble European hockey rinks. When the parameter is rational, the natural map is a homeomorphism, but when the parameter is irrational, the map is not, and in fact, the space-time boundary acquires "fins" (Example 6.3).

There is a weaker version of this class of results, which deal only with the faithful pure space-time harmonic functions, those that come from the interior, and are mapped to the interior of K. Under quite weak conditions, the analogue of this holds even in the $\mathbb{L}^\infty$ case mentioned above; it says that every faithful extremal space-time harmonic function can be identified with a point of the underlying Euclidean space, if the measure is merely absolutely continuous (Theorem 2.4).

When the strong form of the result holds, that the relevant ordered algebra exists and the map is a homeomorphism, there is a type of "ergodic" theorem that equates two norms, one involving space-time, the other concerned with space alone (Theorem 5.1).

Related to this is my original motivation for studying the problem. In the discrete case, the following problem arose from calculations of ordered equivariant K-theory. Decide, given a real polynomial (in several variables) f, and a polynomial with no negative coefficients P, whether there will exist an integer N such that the product $P^N f$ will itself have no negative coefficients. A complete solution is given in [H3]. In the case of a diffuse measure μ (which corresponds to the polynomial P), the analogous question is, given a signed real measure ν (possibly with $\mathbb{L}^\infty$ or continuous Radon–Nikodym derivative with respect to μ), decide whether there is an integer N such that the convolution product, $\mu * \mu * \cdots * \mu * \nu$ (with μ appearing N times) is a measure, that is, nonnegative. In all situations, the space-time boundary gives useful (necessary) information about this, and in some cases (e.g., Proposition 5.2), it even gives sufficient conditions. However, the diffuse situation is far more complicated than the discrete case.

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1. Definitions

There are various boundaries attached to random walks and a proliferation of definitions. The definitions we give below are motivated by the study of dimension groups, and correspond in the discrete case to the Choquet boundary of the dimension group associated to a finitely supported iid on $\mathbf{Z}^d$. We begin with a definition in the discrete state space setting, with an added parameter, an initial distribution. The point of view is that of traces on AF C*-algebras, equivalently on their corresponding Bratteli diagrams [EHS], [VK], ...

Definition of space-time boundary, discrete group. Let X be a (discrete) group, and let $\{\mu_1, \mu_2, \dots\}$ be a family of probability measures on X. Let μ_0 be an initial distribution on X, i.e., a positive but not necessarily finite measure thereon. The convolution of measures μ and ν will be denoted $\mu * \nu$. Define the **space-time cone** determined by $\mu_0, \mu_1, \dots,$

$$\mathcal{C} = \{(x,n) \in X \times \mathbf{N} \text{ such that } \mu_n * \mu_{n-1} * \cdots * \mu_0(\{x\}) > 0\}.$$

As one would expect, this cone consists of the points (x,n) in $X \times \mathbf{N}$ such that x can be reached from a point in the support of μ_0 at *exactly* time n. A **space-time harmonic function** on the cone $\mathcal{C}$ is a nonzero function $f\colon \mathcal{C} \to \mathbf{R}^+$ satisfying the following property:

$$f(x,n) = \sum_{y \in X} f(x+y, n+1)\mu_{n+1}(\{y\}),$$

that is,

$$f(-,n) = f(-,n+1) * \mu_{n+1};$$

"space-time" is often dropped, if no confusion will result between these and spatial (that is, time-independent) harmonic functions defined only for true random walks.

The set of space-time functions forms a convex cone, and is closed in the topology of pointwise convergence. The set of its extreme rays is the **space-time boundary** of the random walk associated to $\{\mu_0, \mu_1, \dots\}$. The bounded space-time harmonic functions form a face of the convex cone; if μ_0 happens to be the counting measure, then the set of extreme rays of bounded harmonic functions is the **Poisson boundary,** here given in the topological setting, rather than the more usual measure-theoretic one. If supp μ_0 is finite, we may normalize harmonic functions, e.g., so that $\sum_{x \in \operatorname{supp} \mu_0} f(x,0) = 1$. In this case, the set of normalized space-time harmonic functions is a Choquet simplex (when equipped with the relative product topology), and the space-time boundary is the extremal boundary of this simplex. Of course, in this definition, only the support of μ_0 matters, not its actual distribution.

In this context, the usual definition of space-time boundary has $\mu = \mu_1 = \mu_2 = \cdots$ (iid), and μ_0 is either a point mass or μ itself (both lead to the same definition). Even if μ has the property that the set of differences, $\operatorname{supp} \mu - \operatorname{supp} \mu$, generates X as an abelian group (this is stronger than $\operatorname{supp} \mu$ itself generating X—consider $X = \mathbf{Z}$ and $\operatorname{supp} \mu = \{1\}$—but the two notions agree if zero belongs to $\operatorname{supp} \mu$), the space-time boundary is sensitive to changes in μ_0. (An elementary iid example on $\mathbf{Z}^2$ is given in [H4; Remark, pp. 46–47] wherein the space-time boundary is compact and connected if the support of μ_0 is a singleton, but neither compact nor connected for certain choices of μ_0 with two point support.)

Note that we implicitly assumed the group operation is abelian; with care, the definitions work in the noncommutative case as well. However, we will not be discussing random walks on nonabelian groups here. One could permit the μ_i to be positive measures (not necessarily finite); we could allow the group X to be replaced by a discrete state space, even varying in discrete time as well. However, these generalizations are not necessary for the purposes of this paper.

Attempting to extend this definition directly to permit diffuse measures on locally compact (abelian) groups runs into problems, because one would like the space-time boundary arising from an iid random walk with μ_0 a point mass to be compact, rather than just an open subset of a compact set. The latter is what happens if one considers only harmonic *functions* on the space-time cone. Instead we have to deal with harmonic *measures*. The motivation for the following construction comes directly from the theory of dimension groups [EHS]. We only extract little bits of the theory for our purposes here.

We temporarily return to the discrete case. Let

$$S_n = \operatorname{supp} \mu_0 * \mu_1 * \cdots * \mu_n;$$

this is just the set of sums $\sum_{i=0}^{n} \operatorname{supp} \mu_i$. Since this is only to motivate the definition to be given shortly, we assume that S_n is finite for all n. Form the finite dimensional vector spaces $\mathbf{R}^{S_n}$; we note that we can identify them with the space of signed measures on S_n, $\mathcal{M}(S_n)$, and moreover, there are linear maps, $*\mu_n : \mathcal{M}(S_n) \to \mathcal{M}(S_{n+1})$ given by convolution with μ_n. We observe that each $\mathcal{M}(S_n)$ is a partially ordered vector space (with positive cone given by the positive measures) and the maps μ_n are positive, i.e., they send positive measures to positive measures. Thus one may form the direct limit as partially ordered vector spaces,

$$H = \lim \mathcal{M}(S_0) \xrightarrow{*\mu_1} \mathcal{M}(S_1) \xrightarrow{*\mu_2} \mathcal{M}(S_2) \xrightarrow{*\mu_3} \mathcal{M}(S_3) \xrightarrow{*\mu_4} \cdots$$

The direct limit consists of pairs (ν, k) where ν is a signed measure supported on S_k, modulo the relation $(\nu, k) \equiv (\nu', k')$ if there exists $l > k, k'$ such that

$$\mu_l * \cdots * \mu_{k+1} * \nu = \mu_l * \cdots * \mu_{k'+1} * \nu'.$$

Equivalence classes are denoted $[\nu, k]$. An element, u, of a partially ordered abelian group G is called an **order unit** if for all g in G, there exists a positive integer N such that $g \leq Nu$. If $\operatorname{supp} \mu_0$ is finite, it is immediate that H admits an order unit, namely $[\mu_0, 0]$. This follows from the simple observation that if $g = [\nu, k]$, there exists equivalent (ν', n) such that

$$d\nu = f\, d\mu^{(n)} \quad \text{where } f \in \mathbb{L}^\infty(X, \mu^{(n)}),$$

so $|\nu| \leq M\mu^{(n)}$ for some M, and $u = [\mu_0, 0] = [\mu^{(n)}, n]$.

The positive cone of H consists of equivalence classes that contain a pair (ν, k) where ν is positive. Of course, dimension group afficionados will recognize this as a special case of the "real" version of dimension groups. A linear function $\tau : H \to \mathbf{R}$ is called a **trace** of H (so-called because of its close connection to traces on AF C*-algebras) if it is positive and nonzero (earlier work in dimension groups uses the terms "state" and "state space" for trace and trace space—however, the former usages would certainly cause confusion). Since we have assumed $\operatorname{supp} \mu_0$ is finite, H will have an order unit, and we may normalize traces, if necessary. From a trace on H, we can construct a space-time harmonic function, via

$$f(x, n) = \tau([\delta_x, n])$$

where δ_x is the point mass at x. Being a linear functional on H entails compatibility conditions on the sequence of linear maps $\tau_n : \mathcal{M}(S_n) \to \mathbf{R}$ obtained by composing τ with the map $\mathcal{M}(S_n) \to H$, and these compatibility conditions are precisely what is required in the definition of space-time harmonic. Conversely, given a space-time harmonic function on the random walk with measures $\mu_0, \mu_1, \mu_2, \ldots$, a unique trace on H can be constructed from it (this is a little more subtle, requiring the Riesz decomposition property), and this trace in turn restricts to the original harmonic function.

The upshot of this construction is that at least in the discrete case, harmonic functions can be identified with certain positive linear functionals on a direct limit of ordered vector spaces. We can now give the appropriate definition of space-time harmonic measure and space-time boundary, in the general case.

Definition of space-time boundaries, general case. Here X is a locally compact (abelian) group, μ is a positive measure on X, $\mathcal{M}(X)$ is the algebra of signed measures on X, $\mathbb{L}^1(X,\mu)$, and $\mathbb{L}^\infty(X,\mu)$ are the usual real $\mathbb{L}^1$ and $\mathbb{L}^\infty$ spaces, and finally $C(X,\mu)$ is the image of the space of bounded continuous real valued functions on X in $\mathbb{L}^\infty(X,\mu)$, that is, the equivalence classes of continuous functions modulo those that vanish on $\operatorname{supp}\mu$. If the support of μ is a subset Y of X, we may also use the notation, $\mathcal{M}(Y)$, $\mathbb{L}^1(Y,\mu)$, $\mathbb{L}^\infty(Y,\mu)$, and $C(Y,\mu)$, respectively. For the purposes of the operations that we are going to perform on the elements of these spaces, we regard them all as spaces of signed measures; thus, $\mathbb{L}^1(X,\mu)$ consists of all signed measures that are absolutely continuous with respect to μ, $\mathbb{L}^\infty(X,\mu)$ consists of signed measures that have bounded (a.e.) Radon–Nikodym derivative with respect to to μ, and $C(X,\mu)$ consists of signed measures which have bounded Radon–Nikodym derivatives with respect to μ that are continuous on the support of μ.

Now let $\{\mu_1, \mu_2, \ldots\}$ be a sequence of probability measures on X, and let μ_0 be a positive measure. Define the convolution powers, $\mu^{(n)} = \mu_0 * \mu_1 * \cdots * \mu_n$, and set $Z_n = \operatorname{supp}\mu^{(n)}$. Of course

$$Z_n = \left\{ \sum_{i=0}^{n} x_i \mid x_i \in \operatorname{supp}\mu_i \right\},$$

and if μ_i are all equal, then $Z_n = (n+1)Z_0$ (the set of all sums of $n+1$ elements of Z_0). We will define three ordered vector spaces, which in the iid case also happen to be algebras, whose sets of extremal traces will be candidates for the space-time boundary, and in each case, this agrees with the formulation in the discrete case. The first two, involving measurable or $\mathbb{L}^1$ functions are much too massive to calculate, and the third, involving $\mathbb{L}^\infty$ functions is much more appropriate.

It turns out that even in the iid situation, this is still too large, and there is a fourth candidate for the space-time boundary, arising from continuous functions. However, it is not always well-defined! It is well-defined in many cases of interest, and it is in this setting that the precise generalization of the finitely supported case can be proved.

For a subset Z of X, and a measure μ with support Y, there is a natural map $\mathcal{M}(Z) \to \mathcal{M}(Z+Y)$ given by $\nu \mapsto \mu * \nu$. This is of course positive and linear. Moreover if Z comes with a measure μ_Z, the map restricts to maps $\mathbb{L}^1(Z,\mu_Z) \to \mathbb{L}^1(Y+Z,\mu*\mu_Z)$ and $\mathbb{L}^\infty(Z,\mu_Z) \to \mathbb{L}^\infty(Y+Z,\mu*\mu_Z)$ (recall that we regard $\mathbb{L}^1$ and $\mathbb{L}^\infty$ as spaces of signed measures). In these cases, the underlying spaces (Z and $Y+Z$) containing the supports of the measures could be replaced by X, for simplicity. In particular, we obtain three direct limi' -ector spaces:

$$\begin{array}{llllllll} G_{\mathcal{M}} & = \lim & \mathcal{M}(Z_0) & \xrightarrow{*\mu_1} & \mathcal{M}(Z_1) & \xrightarrow{*\mu_2} & \mathcal{M}(Z_2 & \xrightarrow{*\mu_3} \dots \\ G_{\text{ac}} & = \lim & \mathbb{L}^1(X,\mu^{(0)}) & \xrightarrow{*\mu_1} & \mathbb{L}^1(X,\mu^{(1)}) & \xrightarrow{*\mu_2} & \mathbb{L}^1(X,\mu^{(2,)}) & \xrightarrow{*\mu_3} \dots \\ G_\infty & = \lim & \mathbb{L}^\infty(X,\mu^{(0)}) & \xrightarrow{*\mu_1} & \mathbb{L}^\infty(X,\mu^{(1)}) & \xrightarrow{*\mu_2} & \mathbb{L}^\infty(X,\mu^{(2)}) & \xrightarrow{*\mu_3} \dots \end{array}$$

In each case, we consider the collection of traces, that is, the positive linear functionals, on the limit ordered vector space together with the zero functional. These are convex cones that can be empty or enormous simplex spaces ([AE]), and the sets of extremal rays are candidates for the space-time boundary. The fourth candidate, when it is defined, is much more tractable, and considerably smaller.

Suppose for each n, that the range of $C(X,\mu^{(n)})$ under convolution with μ_{n+1} lies in the subspace of $\mathbb{L}^\infty(X,\mu^{(n+1)})$, $C(X,\mu^{(n+1)})$. This amounts to the following condition: For all positive measures ν with $d\nu = f\,d\mu^{(n)}$ where f is continuous on Z_n, then we can write $d(\mu*\nu) = g\,d(\mu^{(n+1)})$ where $g: Z_{n+1} \to \mathbf{R}^+$ is continuous. This need not hold, even in the iid case (and with μ_0 a point mass). However, it does hold in a variety of situations. When it does hold, we may form the limit ordered vector space,

$$G_{\text{cont}} = \lim C(X,\mu^{(0)}) \xrightarrow{*\mu_1} C(X,\mu^{(1)}) \xrightarrow{*\mu_2} C(X,\mu^{(2)}) \xrightarrow{*\mu_3} \cdots .$$

Then the space-time boundary of the random walk can be defined as the set of extremal rays of traces on G_{cont}, and it is in this context that we show a version of the weighted moment mapping theorem often holds for diffuse measures.

Now we drop the pretense of generality, and make the following assumptions. First, we assume that X is Euclidean space, i.e., a copy of $\mathbf{R}^d$, and the probability measures, μ_i, are all equal to each other, say to μ for $i \geq 1$ (so the random walk is an iid). We also assume that the initial distribution, μ_0, is either μ itself or a point mass. We next assume that the support of μ is compact. This includes the case of the original weighted moment mapping theorem, wherein μ is finitely supported on a lattice. However, in full generality, and even with some nonsingularity assumptions, there is no guarantee that G_{cont} even exists (Example 6.1). We may make some simple observations.

All of the ordered vector spaces, $G_{\mathcal{M}}$, G_{ac}, G_∞ are partially ordered commutative algebras with no zero divisors. Moreover, the extremal traces of G_∞ are precisely the positive real algebra homomorphisms, and can be naturally identified with the real maximal ideal space (equipped with the point-open, i.e., the weak, topology—this will be discussed in more detail below).

We note the inclusions $G_{\mathcal{M}} \supseteq G_{\text{ac}} \supseteq G_\infty \supseteq G_{\text{cont}}$ (when the last is defined), and the relative ordering on each one agrees with its intrinsic ordering. The multiplication operation on $G_{\mathcal{M}}$ is simply $[\nu, k][\nu', k'] = [\nu * \nu', k + k']$. It is routine to verify that this is well-defined and associative, and obviously products of positive elements are still positive. Moreover, both G_{ac} and G_∞ are closed under this operation. The convolution of two nontrivial signed measures on Euclidean space (or more generally on a torsion free abelian group) is never zero, and it follows easily that no zero divisors exist in $G_{\mathcal{M}}$, hence in none of its subalgebras.

We observed earlier that the element $u = [\mu_0, 0]$ is an order unit for G_∞ and G_{cont} (when the latter exists). Thus no trace can kill u; so we may normalize the traces on G_∞ and G_{cont} so that $\tau(u) = 1$, and the normalized traces form a Choquet simplex. We also observe that u is the multiplicative identity of G_∞. It follows from an elementary argument (in the probabilistic literature, dating back at least to Doob, Snell, and Williamson [DSW]; it appears in work of Gelfand and Naimark in the 1930's), that the extremal traces are precisely the multiplicative ones, see [H1; Theorem I.1], [VK]. Since they are real-valued, we can identify them with the real maximal ideals of the algebra.

All of this applies to G_{cont} as well, provided it is closed under the operation. We know of no instances where G_{cont} is defined, but the multiplication is not; nonetheless, for the multiplication to be defined there is a formally more stringent condition, which we will verify in a class of examples. We fix the definitions of $G_\infty(\mu)$ and $G_{\text{cont}}(\mu)$ to be the respective limits wherein μ_0 is the point mass at the origin, and $\mu_1 = \mu_2 = \cdots = \mu$. The set of extremal or pure traces are denoted $\mathrm{T_e}(G_\infty(\mu))$ and $\mathrm{T_e}(G_{\mathrm{cont}}(\mu))$, respectively. These are two of the possible choices for the space-time boundary of the random walk associated to repeated convolution with μ. In this paper, we are only able to obtain results about the latter.

The algebras $G_{\mathcal{M}}$ and G_{ac} generally do not admit any order units at all, let alone admit the multiplicative identity as an order unit. In these cases, the identification of extreme traces with multiplicative positive functionals may not hold. However, there are some situations wherein this occurs; then the trace space is a proper subset of that of G_∞.

In any event, we are now in a position to define a map Γ^∞ sending the extremal (normalized) traces of G_∞ to the convex hull of the support of μ, $K = \mathrm{cvx\ supp}\,\mu$. Our assumption that $\mathrm{supp}\,\mu$ is compact ensures that K is compact. When G_{cont} is defined, Γ^∞ restricts to a map, $\Gamma\colon \mathrm{T_e}(G_{\mathrm{cont}}) \to K$. The extension of the weighted moment mapping theorem that we are heading for, is that under mild assumptions on μ, $G_{\mathrm{cont}} \equiv G_{\mathrm{cont}}(\mu)$ is defined, is a partially ordered algebra, and Γ is a homeomorphism. If μ is finitely supported on a lattice, Γ "is" the weighted moment map. Moreover, it will be seen that Γ^∞ is an extension of the Legendre transform, in a sense that will be made clear.

Let ν be a compactly supported signed measure on $\mathbf{R}^d$. Associate to it the real analytic function, $P_\nu\colon\mathbf{R}^d \to \mathbf{R}$, defined via

$$P_\nu(r) = \int_{\mathbf{R}^d} \exp(r\cdot w)\, d\nu(w).$$

Of course, this is a real slice of the Fourier transform of ν (sometimes called a Laplace transform, but the latter usually refers to integrals over proper cones in $\mathbf{R}^d$), and so $P_\nu P_{\nu'} = P_{\nu*\nu'}$. We note that if ν is nontrivial and positive, then P_ν vanishes nowhere, and is positive as a function; in particular, this applies to P_μ. Let $\mathcal{A}(\mathbf{R}^d)$ denote the algebra of bounded real analytic functions on $\mathbf{R}^d$. There is a natural map $\mathcal{F}_\mu\colon G_\infty \to \mathcal{A}(\mathbf{R}^d)$ defined by

$$\text{(F)} \qquad \mathcal{F}_\mu[\nu,k](r) = \frac{P_\nu(r)}{P_\mu^k(r)}.$$

The convolution formula is all that is needed to show that $\mathcal{F}_\mu$ is well-defined, and an algebra homomorphism that preserves the identity element. Moreover, positive elements of G_∞ are sent to strictly positive functions of $\mathcal{A} \equiv \mathcal{A}(\mathbf{R}^d)$.

The kernel of $\mathcal{F}_\mu$ is trivial. It is frequently convenient to work with elements of $\mathcal{A}$ of the form P_ν/P_μ^k rather than the original element $[\nu,k]$ of G_∞.

We notice that
$$\frac{\partial P_\mu}{\partial x_i} = \int_K w_i \exp(r\cdot w)\, d\mu(w).$$
Obviously, the (signed) measure given by $d\nu(w) = w_i\, d\mu$ lies in $C(\mathbf{R}^d,\mu)$, so that the element of $\mathcal{A}(\mathbf{R}^d)$ given as $\frac{\partial P_\mu}{\partial x_i}/P_\mu$ belongs to the image of $\mathcal{F}_\mu$. This finally permits us to define Γ^∞. Let τ be an extremal trace on G_∞, normalized so that $\tau([\mu_0,0]) = 1$; we also think of τ as being defined on elements in the image of $\mathcal{F}_\mu$. Set

$$\Gamma^\infty(\tau) = \left(\tau\Big(\frac{\partial P_\mu}{\partial x_1}\Big/P_\mu\Big), \tau\Big(\frac{\partial P_\mu}{\partial x_2}\Big/P_\mu\Big), \dots, \tau\Big(\frac{\partial P_\mu}{\partial x_d}\Big/P_\mu\Big)\right) \in \mathbf{R}^d.$$

We check that the range of Γ^∞ lies in K, the convex hull of the support of μ.

The natural map, $\mathrm{L}^\infty(\mathbf{R}^d,\mu) \to G_\infty$ (at the first level of the direct limit) is obviously order preserving, and sends μ to the identity element, $[\mu,1] = [\mu_0,0]$, assuming we fix μ_0 to be the point mass at the origin. It follows easily that if $d\nu = f\,d\mu$ where f is essentially bounded, then $|\tau([\nu,1])| \le \|f\|_\infty$. Let $u = (u(i))$ be an arbitrary point in $\mathbf{R}^d$, and set $f(x_1,\dots,x_d) = \sum_i u(i)x_i$ and define ν via $d\nu = f\,d\mu$. Thus $|\tau([\nu,1])| \le \max\{|u\cdot w| \text{ such that } w \in K\}$. By shifting K if necessary, and exploiting its convexity and compactness, we can find a collection of vectors $\{u\}$ such that $K = \{\, w \in \mathbf{R}^d \mid u\cdot w \le \max\{\, u\cdot w \mid w\in K\}\,\}$. For any one of these vectors u we calculate,

$$\begin{aligned}|u\cdot\Gamma^\infty(\tau)| &= \left|\sum_{i=1}^d u(i)\tau\Big(\frac{\partial P_\mu}{\partial x_i}\Big/P_\mu\Big)\right| \\ &= \left|\tau\Big(\sum_{i=1}^d u(i)\frac{\partial P_\mu}{\partial x_i}\Big/P_\mu\Big)\right| \\ &= |\tau([\nu,1])| \\ &\le \max\{\, u\cdot w \mid w\in K\}\,.\end{aligned}$$

Allowing u to vary over the prescribed set, we deduce that $\Gamma^\infty(\tau)$ lies in K.

Of course, when the set of extremal traces is equipped with the natural point-open (weak) topology, Γ^∞ is continuous.

We resurface to give some examples of extremal traces. For every point r in $\mathbf{R}^d$, the map, $[\nu,k] \mapsto P_\nu(r)/P_\mu^k(r)$ is obviously multiplicative, as it factors through the point evaluation map $\mathcal{A}(\mathbf{R}^d) \to \mathbf{R}$, sending a real analytic function to its value at r. If ν is a positive measure, $[\nu, k]$ is sent to a positive (not just nonnegative) real number; in particular, this is a trace on G_∞, denoted τ_r; this is a **point evaluation trace.** In particular, when $r = 0$, the effect of the point evaluation trace is to send the signed measure to its barycentre, normalized for the fact that the measure lives on kK, rather than K. We have a map $\mathbf{R}^d \to K$ given by $r \mapsto \Gamma^\infty(\tau_r)$. It is easy to verify that this map has image in the interior of K, $\operatorname{Int}(K)$, and is just $r \mapsto (\nabla \ln P_\mu)(r)$. This is an example of the Legendre transform of convex analysis. We observe that point evaluation traces are examples of *faithful* traces (a trace τ on a partially ordered vector space, G, is **faithful** if $\ker \tau \cap G^+ = \{0\}$).

A theorem, due to Rothaus [Rh], asserts that with the appropriate conditions on the measure μ, the map $\nabla \ln P_\mu : \operatorname{Int}(K) \to \mathbf{R}^d$ is a homeomorphism; this is extended in the book by Rockafeller [Ro; Theorem 26.1]. Even if $\nabla \ln P_\mu$ is a homeomorphism, it does not follow that the restriction of Γ^∞ to the pure faithful traces is one to one (Example 2.5).

In the case that μ is finitely supported on the lattice in $\mathbf{Z}^d$, we can recover the weighted moment map of algebraic geometry. When μ has finite support in a lattice, we can regard P_ν as polynomials in exponentials; thus, identifying the monomial x^w with $e^{\ln x \cdot w}$ (for x in $\mathbf{R}^d$ having strictly positive coordinates), the range of $\mathcal{F}_\mu$ will consist of certain rational functions. In particular, $P_\mu = \sum \lambda_w x^w$ (where $\lambda_w = \mu(\{w\})$), and $\Gamma^\infty(\tau) = \sum_{\operatorname{supp} \mu} \lambda_w \tau(x^w/P_\mu) w$, and restricted to point evaluations at $s = \exp(r)$ (a strictly positive d-tuple), the map is $s \mapsto \sum \lambda_w (s^w/P_\mu(s)) w$, which is the weighted moment map appearing in [Od; p. 94]. See also [H4], [H6; pp. 58–61], and [H8].

In this case, the extremal traces consist of actual (space-time) harmonic functions. If instead the measure is diffuse, not all traces come from functions (continuous or otherwise) on subsets of $\mathbf{R}^d \times \mathbf{N}$. Instead, they come from measures. Explicitly, restrict τ to the image of the first level of $\mathbb{L}^\infty(\mathbf{R}^d, \mu)$; this defines a positive linear functional thereon; if instead, $G_{\text{cont}}(\mu)$ is defined, we obtain a positive linear functional on $C(K)$, hence a measure on K. These are not all obtainable from functions. Instead the compatibility conditions inherent in traces on the limit serve to define (space-time) harmonic measures.

In the next section ("Faithful traces"), we shall show that under fairly weak conditions, the faithful pure traces on $G_\infty(\mu)$ are all point evaluations, and the same conclusions apply with $G_{\text{cont}}(\mu)$ when it exists. This may be viewed as a weak generalization of the result in the finitely supported case, corresponding to the interior. (With absolutely no conditions on μ, this fails, Example 2.5.) The following section deals with sufficient conditions in order that $G_{\text{cont}}(\mu)$ exist; the case of least difficulty occurs when $K = \text{cvx supp}\,\mu$ is strictly convex; more general results are also obtained. The section, "Perfidious traces", gives conditions under which the precise analogue of the weighted moment mapping theorem holds, namely that $G_{\text{cont}}(\mu)$ exist and $\Gamma\colon \mathrm{T_e}(G_{\text{cont}}(\mu)) \to K$ be a homeomorphism. For example, this occurs if μ is absolutely continuous and K is strictly convex, or if K is a polytope and μ is absolutely continuous with Radon–Nikodym derivative $h\colon K \to \mathbf{R}$ that is continuous and does not vanish at any vertex.

Other results, also allowing higher dimensional faces in the boundary, are obtained; however, these are not as strong. The fifth section discusses a consequence of Γ being a homeomorphism, specifically that the point evaluation traces are dense in the pure trace space; this is equivalent to a condition resembling the ergodic theorem. Section 6 contains a couple of examples, together with a general result on planar convex sets.

2. Faithful traces

In this section, we show that if μ is a compactly supported (Borel) probability measure on $\mathbf{R}^d$ that is not singular, then every faithful pure trace on $G_\infty(\mu)$ is a point evaluation. By "not singular", we mean that the measure has a nonzero absolutely continuous part. In addition, if $G_{\text{cont}}(\mu)$ exists (and μ still satisfies the other assumptions), all pure faithful traces on $G_{\text{cont}}(\mu)$ are also just point evaluations. In particular, in the topology of pointwise convergence, the set of faithful pure traces is naturally homeomorphic to $\mathbf{R}^d$, and to $\text{Int}(K)$, the latter by means of the Legendre transformation. On the other hand, it is quite easy to construct singular measures μ for which K contains a d-ball, for which the point evaluations do not exhaust the faithful pure traces on $G_{\text{cont}}(\mu)$ and $G_\infty(\mu)$, Example 2.5.

Recall that a faithful pure trace on $G = G_\infty(\mu)$ (or $G = G_{\text{cont}}(\mu)$) is a positive, multiplicative homomorphism, $\tau\colon G \to \mathbf{R}$ such that $\tau(g) > 0$ for all g in G^+. In this case, G has been identified with a subalgebra of $\mathcal{A}(\mathbf{R}^d)$, and so has no divisors of zero. Set $\Pi = G^+ \setminus \{0\}$ (if we must specify that $G = G_{\text{cont}}$, then the corresponding set of nonzero positive elements will be denoted Π_c). Then Π is multiplicatively closed (that is, a product of any two elements of Π belongs to Π), and we may form the commutative algebra $G[\Pi^{-1}]$, obtained by inverting every element of Π. (This is "localization" from elementary commutative algebra.) We may impose a partial ordering on $G[\Pi^{-1}]$, by declaring a fraction $ab^{-1} \geq 0$ for a in G and b in Π if there exists c in Π such that $ac \in G^+$. Since $ab^{-1} = (ac)(bc)^{-1}$, it follows that $G[\Pi^{-1}]$ is a partially ordered algebra with this ordering.

Unfortunately, $1 = [\mu_0, 0]$ is not an order unit for $G[\Pi^{-1}]$. However, we are really only interested in the multiplicative positive homomorphisms on $G[\Pi^{-1}]$.

We first observe that every faithful pure trace on G extends uniquely to a multiplicative positive homomorphism on $G[\Pi^{-1}]$, via $\overline{\tau}(ab^{-1}) = \frac{\tau(a)}{\tau(b)}$ (the point being that $\tau(b) > 0$). The next step is to show that every positive multiplicative linear functional on $G[\Pi^{-1}]$ is of the form $ab^{-1} \mapsto \mathcal{F}_\mu(a)(r_0)/\mathcal{F}_\mu(b)(r_0)$ for some r_0 in $\mathbf{R}^d$ (see equation (F) of section 1 for the definition of $\mathcal{F}$).

To this end, we define the algebra, $\mathcal{L} \equiv \mathrm{L}^\infty_{\mathrm{cct}}(\mathbf{R}^d)$, consisting of all signed (Borel) measures ν with compact support in $\mathbf{R}^d$ that are absolutely continuous with respect to λ (Lebesgue measure) with bounded Radon–Nikodym derivative. Multiplication is by convolution. With the natural ordering ($\mathcal{L}^+$ consisting of nonnegative measures), $\mathcal{L}$ is a partially ordered commutative algebra. Let Π_∞ denote $\mathcal{L}^+\backslash 0$, and form $\mathcal{L}[\Pi_\infty^{-1}]$, the localized algebra. As with $G[\Pi^{-1}]$, this is a partially ordered algebra. Define the algebra $\mathcal{L}^1_{\mathrm{cct}}$ obtained from the signed measures with compact support that are absolutely continuous with respect to to Lebesgue measure; define as well $\Pi_1 := \mathcal{L}^1_{\mathrm{cct}} \setminus \{0\}$, and the localization, $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$. We shall show that there is a natural identification of $G[\Pi^{-1}]$ with $\mathcal{L}[\Pi_\infty^{-1}]$ and $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$ in such a way that their orderings coincide. Then we show that every positive multiplicative linear functional on $\mathcal{L}[\Pi_\infty^{-1}]$ arises from a point evaluation (in the appropriate sense).

Let $\mathcal{A}_{\mathrm{u}}(\mathbf{R}^d)$ denote the algebra of real analytic functions on $\mathbf{R}^d$; obviously, $\mathcal{A}(\mathbf{R}^d)$ is a subalgebra (with respect to pointwise multiplication). For compactly supported signed measures ν, recall that the assignment $\nu \mapsto P_\nu$ (where $P_\nu = \int_{\mathbf{R}^d} \exp(r \cdot w)\, d\nu(w)$), is multiplicative (with respect to to convolution). Since this map sends Π_1 to strictly positive functions in $\mathcal{A}_{\mathrm{u}}(\mathbf{R}^d)$, it extends to an algebra embedding, $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}] \to \mathcal{A}_{\mathrm{u}}(\mathbf{R}^d)$. Moreover, the image of G under $\mathcal{F}_\mu$ is obviously contained in the image of $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$, and thus $\mathcal{F}_\mu$ extends to an algebra embedding $G[\Pi^{-1}] \to \mathcal{A}_{\mathrm{u}}(\mathbf{R}^d)$, whose image is contained in that of $\mathcal{L}[\Pi_\infty^{-1}]$. It is clear that the orderings and the multiplications coincide. It remains to show the image of $G[\Pi^{-1}]$ is all of $\mathcal{L}^1_{\rm cct}[\Pi_1^{-1}]$; we do this by showing the former contains $\mathcal{L}[\Pi_\infty^{-1}]$, and $\mathcal{L}[\Pi_\infty^{-1}]$ contains $\mathcal{L}^1_{\rm cct}[\Pi_1^{-1}]$. Of course, it is at this point that we must hypothesize something about μ.

LEMMA 2.1: *Suppose that μ is a compactly supported probability measure on $\mathbf{R}^d$ that is not singular with respect to Lebesgue measure. Setting $G = G_\infty(\mu)$, we have $G[\Pi^{-1}] = \mathcal{L}[\Pi_\infty^{-1}] = \mathcal{L}^1_{\rm cct}[\Pi_1^{-1}]$. If $G_{\rm cont}(\mu)$ exists, then $G_{\rm cont}(\mu)[\Pi_c^{-1}] = \mathcal{L}^1_{\rm cct}[\Pi_1^{-1}]$.*

Proof: For any compactly supported probability measure ν, we define $S_\nu = G_\infty(\nu)[\Pi(\nu)^{-1}]$; for C a compact convex subset (containing a d-ball) of $\mathbf{R}^d$, let λ_C denote the restriction of Lebesgue measure to C. We first show that $S_{\lambda_C} = S_{\lambda_{tC}}$ for all t.

We may assume that the origin lies in the interior of C. For any integer $k > t$, we form $\lambda_C^{(k)}$, the k-fold convolution of λ_C with itself. It is absolutely continuous with respect to λ_{kC}, say with Radon–Nikodym derivative h; in fact, $h: kC \to \mathbf{R}^+$ is continuous, and its zero set is contained in the boundary of kC. Now $t < k$, so there exists $\epsilon > 0$ such that $tC \subseteq h^{-1}([\epsilon, \infty])$. Hence $\lambda_{tC} \leq \frac{1}{\epsilon}\lambda_C^{(k)}$, so that the Radon–Nikodym derivative of the former with respect to the latter is bounded. Thus $P_{\lambda_{tC}}/P^k_{\lambda_C} = \mathcal{F}_{\lambda_C}[\lambda_{tC}, k]$ is the image of an element of $G_\infty(\lambda_C)$. Hence λ_{tC} belongs to S_{λ_C}. It follows immediately that any signed measure ν in $\mathcal{L}^\infty(tC, \lambda_{tC})$ belongs to S_{λ_C}. Since we can obviously replace t by $1/k$ and k by $1/c$, the reverse inclusion follows.

In fact, the argument of the preceding paragraph shows that if μ_i are positive measures and $\mu_1 \leq M\mu_2^{(k)}$ for some positive integer k and positive real number M, then $S_{\mu_1} \subseteq S_{\mu_2}$.

Now decompose $\mu = \mu_{\rm s} + \mu_{\rm ac}$ into its singular and absolutely continuous parts; the hypothesis is that the latter is not zero. Let $K = \operatorname{cvx} \operatorname{supp} \mu$. Write $d\mu_{\rm ac} = h d\lambda_K$. On writing $d\mu_{\rm ac}^{(3)} = h_3 d\lambda_K^{(3)}$, we find h_3 is continuous. So $h_3^{-1}[\epsilon, \infty]$ contains an open set for some $\epsilon > 0$, and therefore contains a closed d-ball C. Thus $\lambda_C \leq \frac{1}{\epsilon}\mu_{\rm ac}^{(3)}$, and it follows that $S_{\mu_{\rm ac}} \subseteq S_{\lambda_C}$; the reverse inclusion follows from $\mu_{\rm ac} \leq M\lambda^{(tC)}$ for some positive t and M, and $S_{\lambda_C} = S_{\lambda_{tC}}$.

We also have that $\mu_{\rm ac} \leq \mu$, so that $S_{\mu_{\rm ac}} \subseteq S_\mu$. Finally, we notice that $\mu' = \mu * \mu_{\rm ac}$ is absolutely continuous, and thus $S_{\mu'} = S_{\lambda_C}$. Any element of S_μ can be written as $(\nu_1)(\nu_2)^{-1} = (\nu * \mu_{\rm ac})(\nu_2 * \mu_{\rm ac})^{-1}$ (where ν_2 is positive), and we see both numerator and denominator are absolutely continuous (and have compact support, are bounded, etc.), so belong to S_{λ_C} for any closed d-ball C.

The upshot is that $S_\mu = S_{\lambda_C}$ for any closed d-ball C. We now show $\mathcal{L}[\Pi_\infty^{-1}] \subseteq S_{\lambda_C}$, which will complete the proof that $\mathcal{L}[\Pi_\infty^{-1}] = G_\infty[\Pi^{-1}]$. For ν a compactly supported bounded signed measure, choose a d-ball C such that $|\nu| \le M\lambda_C$. By translation, we may assume the origin is contained in the interior of C, so that $\lambda^{(2)} \le \lambda_{2C}$ and $\lambda_C \le 2\lambda_{2C}$. Then $P_\nu = (P_\nu/P_{\lambda_C})P_{\lambda_C}$. However,

$$P_{\lambda_C} = \frac{P_{\lambda_C}^2}{P_{\lambda_{2C}}}\left(\frac{P_{\lambda_C}}{P_{\lambda_{2C}}}\right)^{-1}$$

expresses P_{λ_C} in the form ab^{-1} where each of a and b are in the image of S_{λ_C}, so P_{λ_C} belongs to $\mathcal{L}[\Pi_\infty^{-1}]$, and P_ν/P_{λ_C} is in the image of $G_\infty(\lambda_C)$. This gives an expression for ν as a product of elements of S_{λ_C}, so $\mathcal{L}[\Pi_\infty^{-1}] \subseteq S_{\lambda_C}$ and thus equality holds.

To see that $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}] \subseteq \mathcal{L}[\Pi_\infty^{-1}]$ (the reverse inclusion is trivial), just observe that if ν is absolutely continuous with respect to to μ with support contained in a ball C, then $\nu * \lambda_C$ has bounded Radon–Nikodym derivative with respect to λ_{2C}, so that $P_\nu = (P_{\nu*\lambda_C}/P_{\lambda_{2C}})(P_{\lambda_C}/P_{\lambda_{2C}})^{-1}$, which shows ν belongs to $\mathcal{L}[\Pi_\infty^{-1}]$.

Finally, suppose $G_{\mathrm{cont}}(\mu)$ exists. Observe that if ν is a finite signed measure on $\mathbf{R}^d$ with compact support, then $\nu * \lambda_C$ is absolutely continuous with respect to Lebesgue measure, and $\nu * \lambda_C * \lambda_C$ has continuous derivative, with compact support. This shows $\mathcal{L}^1_{\mathrm{cct}} \subseteq G_{\mathrm{cont}}(\mu)[\Pi_{\mathrm{c}}^{-1}]$, which is enough to obtain $G_{\mathrm{cont}}(\mu)[\Pi_{\mathrm{c}}^{-1}] = \mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$. ∎

Now we have to find the positive and multiplicative linear functionals on $\mathcal{L}^1_{\mathrm{cct}}$. It presumably is well known that they are all of the form

$$\gamma_r \colon \nu \mapsto \int_{\mathbf{R}^d} \exp(r \cdot w)\, d\nu(w) = P_\nu(r) \quad \text{for some } r \text{ in } \mathbf{R}^d,$$

but I have not been able to find a reference. It is of course standard that the multiplicative functionals on the full convolution algebra $\mathrm{L}^1(\mathbf{R}^d)$ are of this form (with r in $\sqrt{-1}\mathbf{R}^d$), but its proof has to be adapted nontrivially. One of the problems is that γ_r is not continuous with respect to to the (global) L^1 norm, except when r is the origin.

To get around this, let C_N denote the closed ball of radius N centred at the origin, and write $\mathcal{L}^1_{\mathrm{cct}} = \cup \mathrm{L}^1(C_N, \lambda|C_N)$ (note the unnormalized $\lambda|C_N$, not

λ_{C_N}). Let γ be a multiplicative positive linear functional on $\mathcal{L}^1_{\rm cct}$, and denote its restriction to $L^1(C_N, \lambda|C_N)$, γ_N. Then the multiplicative property is lost, but it is still a positive linear functional on a *bona fide* L^1-space. The following general and elementary proof of automatic continuity for positive linear functionals on ordered Banach spaces was found by George Elliott, after I had found a very labourious one in the case of L^1-spaces.

PROPOSITION 2.2 (Elliott): *Let B be a partially ordered (real) Banach space with the property that there exists positive real M such that for all b in B, there exists b' in B^+ satisfying $b \leq b'$ and $\|b'\| \leq M\|b\|$. Then any positive linear functional on B is continuous. In particular, this applies if B is an L^1-space.*

Proof: If the positive linear functional α were not continuous, there would exist b_i in the unit sphere of B such that the real numbers, $\alpha_i = \alpha(b_i)$, are positive, increasing and unbounded. By removing enough of them, we may also assume that $\sum 1/\alpha_i$ converges. By hypothesis, there exist b_i' in B^+ such that $b_i \leq b_i'$ and $\|b_i'\| \leq M$. Set

$$c_i = \frac{1}{\alpha_i} b_i';$$

then $\|c_i\| \leq M/\alpha_i$, and thus $\{\sum_{i=1}^n c_i\}$ converges to an element c of B^+. We observe that for all n, $c \geq \sum_{i=1}^n c_i$. Hence

$$\alpha(c) \geq \sum_{i=1}^n \alpha(c_i) = \sum_{i=1}^n \frac{\alpha(b_i')}{\alpha_i} \geq \sum_{i=1}^n \frac{\alpha(b_i)}{\alpha_i} = n.$$

Since this is true for all n, we arrive at a contradiction.

If B is an L^1-space, we may set $b' = |b|$, and then the hypotheses will be satisfied with $M = 1$. ∎

PROPOSITION 2.3: *Every multiplicative positive linear functional on $\mathcal{L}^1_{\rm cct}$ is given by*

$$\nu \mapsto \int_{\mathbf{R}^d} \exp(r \cdot w)\, d\nu(w)$$

for some r in $\mathbf{R}^d$.

Proof: We adapt the standard proof, given in [Ru; p. 207], of the corresponding result (with purely imaginary r) for the full L^1 convolution algebra. For this proof, it is convenient to drop our convention that elements of L^1 spaces be regarded as absolutely continuous measures, and instead revert to the usual formulation as spaces of functions. Let γ be the multiplicative positive functional on $\mathcal{L}^1_{\mathrm{cct}}$, and denote its restriction to $L^1(C_N)$, γ_N. By Proposition 2.2, each γ_N is continuous, so there exists unique β_N in $L^\infty(C_N)$ such that $\gamma(f) = \int_{C_N} f(w)\beta_N(w)\,d\lambda(w)$ for all essentially bounded f supported in C_N. Compatibility of the linear functionals together with uniqueness of each of the β_N ensures that $\beta_{N'}|C_N = \beta_N$ (a.e.) if $N' > N$, and we may define (a.e) a function $\beta\colon \mathbf{R}^d \to \mathbf{R}$ via $\beta|C_N = \beta$. (Naturally, β is almost never bounded, although each of the β_N's is.) It is immediate that β is measurable, and for all functions f in $\mathcal{L}^1_{\mathrm{cct}}$, $\gamma(f) = \int_{C_N} f(w)\beta(w)\,d\lambda(w)$ for all sufficiently large N. Suppose f and g are supported in C_N, so their convolution product, $f * g$ is supported in C_{2N}. For y in C_N, define f_y with support in C_{2N} via $f_y(w) = f(w-y)$. Now

$$\begin{aligned}
\gamma(f*g) &= \int_{\mathbf{R}^d} (f*g)(w)\beta(w)\,d\lambda(w) = \int_{C_{2N}} (f*g)(w)\beta_{2N}(w)\,d\lambda(w) \\
&= \int_{w\in C_{2N}}\int_{y\in C_N} f(w-y)g(y)\,d\lambda(y)\;\beta_{2N}(w)\,d\lambda(w) \\
&= \int_{y\in C_N}\int_{w\in C_{2N}} f_y(w)\beta_{2N}(w)\,d\lambda(w)\;g(y)\,d\lambda(y) \\
&= \int_{C_N} g(y)\gamma(f_y)\,d\lambda(y).
\end{aligned}$$

Thus,

$$\int_{C_N} g(y)\gamma(f_y)\,d\lambda(y) = \gamma(f*g) = \gamma(f)\gamma(g) = \gamma(f)\int_{C_N} g(y)\beta_N(y)\,d\lambda(y).$$

Uniqueness of the representing measure β_N ensures that $\gamma(f)\beta_N(y) = \gamma(f_y)$ (a.e. in y). If we choose $f \geq 0$ but not equal to zero, then $\gamma(f) \neq 0$ (since γ is not identically zero, and positivity of γ entails positivity a.e. of the representing measure β_N, hence of β). Since the equation is true for all N and all positive f in $\mathcal{L}^1_{\mathrm{cct}}$, we have $\beta(y) = \gamma(f_y)/\gamma(f)$. We may replace y by $w+y$ for any w in $\mathbf{R}^d$, and so obtain

$$\beta(w+y) = \frac{\gamma(f_{w+y})}{\gamma(f)} = \frac{\gamma((f_w)_y)}{\gamma(f)} = \frac{\gamma((f_w)\beta(y))}{\gamma(f)} = \beta(w)\beta(y).$$

Since β is measurable, the functional equation $\beta(w+y) = \beta(w)\beta(y)$ (a.e.) forces $\beta(w) = \exp(r\cdot w)$ for some complex d-tuple r. However, since γ is real valued, so must β be and it follows that r is real. ∎

THEOREM 2.4: *Let μ be a compactly supported measure on $\mathbf{R}^d$ that is not singular with respect to Lebesgue measure. Then every faithful pure trace τ on each of $G_{\mathrm{ac}}(\mu)$, $G_\infty(\mu)$, and (if it exists) $G_{\mathrm{cont}}(\mu)$ is a point evaluation; that is, there exists r in $\mathbf{R}^d$ such that the trace is given by*

$$\tau([\nu,k]) = \frac{\int_{\mathbf{R}^d} \exp(r\cdot w)\,d\nu(w)}{\left(\int_{\mathbf{R}^d} \exp(r\cdot w)\,d\mu(w)\right)^k}.$$

Proof: We note that $G_\infty(\mu) \subseteq G_{\mathrm{ac}}(\mu) \subseteq \mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$, so

$$G_\infty(\mu)[\Pi_\infty^{-1}] = G_{\mathrm{ac}}(\mu)[\Pi_{\mathrm{ac}}^{-1}],$$

and this equals $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$. Hence every faithful pure trace on any of the three (or two) algebras extends uniquely to a multiplicative and positive linear functional on $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$. Restricting this to $\mathcal{L}^1_{\mathrm{cct}}$, it is still a multiplicative and positive linear functional, and so is of the form given in Proposition 2.3. Moreover, such homomorphisms are strictly positive on positive measures, and are multiplicative, so extend uniquely to the algebra obtained by inverting the nonzero positive elements, i.e., $\mathcal{L}^1_{\mathrm{cct}}[\Pi_1^{-1}]$. Uniqueness of the extension yields the result. ∎

Example 2.5: We exhibit a very simple class of examples to show that the conclusion of Theorem 2.4 can fail if μ is purely singular. (Notice that if some convolution power of μ is not singular, then Theorem 2.4 still applies.) Let μ be an atomic measure with finite support, S, in $\mathbf{R}^d$. Then $G_{\mathrm{cont}}(\mu)$ is defined and equals $G_\infty(\mu)$; moreover, by cutting down to an affine subspace if necessary, we may assume cvx S contains a d-ball. If S is contained in the canonical copy of of $\mathbf{Z}^d$ in $\mathbf{R}^d$, or more generally, if the subgroup of $\mathbf{R}^d$ generated by the set of differences $S - S = \{\, s - s' \mid s, s' \in S\}$ is discrete (e.g., if $S \subseteq \mathbf{Q}^d$), then the point evaluations from $\mathbf{R}^d$ exhaust the faithful pure traces on $G_{\mathrm{cont}}(\mu)$, and the theorem in the finitely supported case applies. However, if this subgroup is not discrete, then there exist faithful pure traces other than point evaluations, arising from a copy of a larger real vector space.

To illustrate this phenomenon, let $d = 1$ and $S = \{0, 1, \sqrt{2}\}$; suppose the measure μ assigns equal mass to these three points (all that is important is that it assign some to each). Paralleling the development in [H4], $G_{\mathrm{cont}}(\mu)$ can be regarded as a ring of certain rational functions in the variables, x and $x^{\sqrt{2}}$ (replacing $\exp r$ and $\exp(\sqrt{2}r)$). In particular, $G_{\mathrm{cont}}(\mu)$ is the algebra $\mathbf{R}[(1+x+x^{\sqrt{2}})^{-1}, x(1+x+x^{\sqrt{2}})^{-1}, x^{\sqrt{2}}(1+x+x^{\sqrt{2}})^{-1}]$, with positive cone generated additively and multiplicatively by the generators. Now the pure traces arising as point valuations from $\mathbf{R}$, send x to $t \in \mathbf{R}^+ \setminus \{0\}$ and $x^{\sqrt{2}}$ to $t^{\sqrt{2}}$. However, $\{1, \sqrt{2}\}$ is linearly independent over the rational numbers, so $\left\{x, x^{\sqrt{2}}\right\}$ is algebraically independent (over the reals). In particular, we can send x and $x^{\sqrt{2}}$ to any two positive numbers we wish, and still obtain a faithful pure trace. The set of faithful pure traces contains a copy of $\mathbf{R}^2$ (via its exponential, the positive quadrant), and these constitute all of the faithful pure traces. There is a natural isomorphism between $G_{\text{cont}}(\mu)$ and R_P of [H4; Example 1A]. The same process will work for any finite set S, to give an isomorphism between $G_{\text{cont}}(\mu)$ and some algebra of the form R_P, whose pure faithful traces have been completely analyzed.

As a minor variation on this example, let μ be an atomic probability measure with infinitely many points in its support, S. For instance

$$S = \{a_n\}_{n \in \mathbf{N}} \cup \{0\} \subseteq [0,1].$$

Then if $\{a_n\}$ is rationally linearly independent, there are faithful pure traces given by sending $\exp(a_n r)$ to completely arbitrary positive real numbers, z_n, respectively, subject to a normalization condition imposed by convergence of $\sum \mu(\{a_n\})$. The upshot is a cone of faithful pure traces generating a weighted l^1 space. ∎

3. Existence results for $G_{\text{cont}}(\mu)$

Unless μ is atomic, $\Gamma^\infty : \mathrm{T_e}(G_\infty(\mu)) \to K$ is not a homeomorphism, as $C(K)$ is separable; so $G_\infty(\mu)$ is too large. Thus we work with the much smaller algebra, $G_{\text{cont}}(\mu)$. A problem is that for some choices of μ, $G_{\text{cont}}(\mu)$ need not exist, Example 6.1. In this section, we show in particular, that if $K = \text{cvx supp}\,\mu$ is strictly convex (i.e., has no line segments in its boundary), and μ is otherwise (almost) unencumbered, then $G_{\text{cont}}(\mu)$ exists, and is a partially ordered algebra (the latter is crucial when considering its extremal traces). We also obtain some existence results when higher dimensional faces occur in K, but these are far more difficult and technical.

Certainly, $G_{\text{cont}}(\mu)$ as defined in the introduction exists if there exists an integer n such that for all $m \geq n$, whenever $h : mK \to \mathbf{R}$ is continuous and $d\nu = h\,d\mu$ determines ν, then the function $h' : (m+1)K \to \mathbf{R}$ defined by $d(\nu * \mu) = h'\,d\mu^{(m+1)}$

is continuous. This says simply that the map

$$C(mK, \mu^{(m)}) \to \mathrm{L}^\infty((m+1)K, \mu^{(m+1)})$$

given by convolution with μ has range in $C((m+1)K, \mu^{(m+1)})$. (Recall our convention that $C(Y, \nu)$ consists of the signed measures on Y absolutely continuous with respect to ν whose Radon–Nikodym derivatives are continuous, and two continuous functions are equivalent if the zero set of their difference has full measure with respect to μ.) In fact, this is a bit stronger than is necessary for the direct limit to be defined, since telescoping is permitted. In order for it to be a partially ordered algebra, the following condition is sufficient. If $h{:}mK \to \mathbf{R}$ and $k{:}m'K \to \mathbf{R}$ are continuous positive functions (with $m \geq n$) and ν and κ are the corresponding measures, then

$$\text{(3.1)} \qquad \frac{d(\nu * \kappa)}{d\mu^{(m+m')}}{:}(m+m')K \to \mathbf{R}$$

is continuous. (On the face of it, this should have something to do with the work of Guivarc'h [Gu]; exactly what is unclear.)

A proper subset F of a compact convex set $K \subset \mathbf{R}^d$ is called a **face** of K if for all α in the open interval $(0,1)$ and all elements v and w of K, $\alpha v + (1-\alpha)w$ belongs to F if and only if each of v and w do. A point v of K is **extreme** if the set $\{v\}$ is a face of K. A face F of K is **exposed** by u in $\mathbf{R}^d$ (or a linear functional on $\mathbf{R}^d$) if $F = \{v \in K \mid u \cdot v = \max_{w \in K} u \cdot w\}$. An extreme point is **exposed** if the singleton set it constitutes is exposed as a face. A compact convex set is **strictly convex** if all faces are singletons, or what amounts to the same thing (at least in $\mathbf{R}^d$), the boundary contains no line segments. If K is strictly convex, all of its boundary points are exposed [Va; Theorem 7.7, p. 94]. A compact convex subset of $\mathbf{R}^d$ is called a **convex body** if it contains a d-ball.

LEMMA 3.1: *Let K be a compact convex body in $\mathbf{R}^d$ with exposed point v. Let K_1 and K_2 be convex bodies in $\mathbf{R}^d$ such that $K_1 + K_2 = K$. Let v_i be points of K_i such that $v = v_1 + v_2$. Let μ_i be probability measures on K_i with* cvx supp $\mu_i = K_i$, *and set $\mu = \mu_1 * \mu_2$. Suppose that η_i are finite signed measures on K_i, with $d\eta_i = h_i d\mu_i$, where each $h_i{:}K_i \to \mathbf{R}$ is continuous. Then*

$$\big(d(\eta_1 * \eta_2)/d\mu\big)(v) = h_1(v_1)h_2(v_2).$$

Let $\{z_j\}$ be a sequence of points of K converging to v, with the property that μ does not vanish on any neighbourhood of each z_j. Then

$$\lim_j \left(\frac{d(\eta_1 * \eta_2)}{d\mu}\right)(z_j) = h_1(v_1)h_2(v_2).$$

Proof: We first observe that the v_i exist and are unique, and moreover are extreme points of K_i. Let u expose v with respect to K; say, $u \cdot v = 1$ and $u \cdot v' < 1$ for v' in $K \setminus \{v\}$. It follows easily that u also exposes each of v_i relative to K_i. Define

$$U_\epsilon = \{ w \in K \mid u \cdot w > 1 - \epsilon \} ,$$

so that $\bigcap_{\epsilon \to 0} U_\epsilon = \{v\}$. Denote the diameter of a set U, $\operatorname{diam}(U)$ (Euclidean distance is good enough here). Then obviously $\operatorname{diam}(U_\epsilon) \to 0$. We may suppose

$$u \cdot v_i = \alpha_i > 0;$$

then $\alpha_1 + \alpha_2 = 1$ and $u \cdot y < \alpha_i$ for all $y \in K_i \setminus \{v_i\}$. Define the counterparts of U_ϵ,

$$U_\epsilon^i = \{ y \in K_i \mid u \cdot y > \alpha_i - \epsilon \} .$$

Clearly, $\{ (w_1, w_2) \in K_1 \times K_2 \mid w_1 + w_2 \in U \} \subseteq U_\epsilon^1 \times U_\epsilon^2$. As u also exposes v_i, both $\operatorname{diam}(U_\epsilon^i) \to 0$ as $\epsilon \to 0$. By uniform continuity of each of the h_i, for $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ (with $\delta(\epsilon) \to 0$ as $\epsilon \to 0$) such that $|y^1 - y^2| < \delta(\epsilon)$ (where y^j belong to K_i) implies $|h_i(y^1) - h_i(y^2)| < \epsilon$. Since $\operatorname{diam}(U_\epsilon^i) \to 0$ for each i, there exists $f(\delta) \to 0$ such that $U^i_{f(\delta)}$ is contained in the ball of radius $\delta(\epsilon) \equiv \delta$ centred at v_i. Choose $\epsilon > 0$. On $U^i_{f(\epsilon)}$, h_i differs from the constant function with value $h_i(v_i)$ by at most ϵ. If B is an open set containing v and contained in ball centred at v with radius $\delta(\epsilon)$ or less, then we see that

$$\begin{aligned}
|\eta_1 * \eta_2(B) - h_1(v_1)h_2(v_2)\mu(B)| &= \left| \int_{K_2} \eta_1(B-x)\, d\eta_2(x) - \int_{K_2} \mu_1(B-x)\, d\mu_2(x) \right| \\
&= \left| \int_{x \in K_2} \int_{y \in B-x} h_1(y)h_2(x) - h_1(v_1)h_2(v_2)\, d\mu_1(y) d\mu_2(x) \right| \\
&= \left| \int_{x \in U^2_{f(2\delta(\epsilon))}} \int_{y \in (B-x)\cap U^2_{f(2\delta(\epsilon))}} h_1(y)h_2(x) - h_1(v_1)h_2(v_2)\, d\mu_1(y)\, d\mu_2(x) \right| \\
&\leq \epsilon(|h_1(v_1)| + |h_2(v_2)|) \int_{x \in U^2_{f(2\delta(\epsilon))}} \int_{y \in (B-x)\cap U^2_{f(2\delta(\epsilon))}} d\mu_1(y) d\mu_2(x) \\
&= \epsilon(|h_1(v_1)| + |h_2(v_2)|)\mu(B).
\end{aligned}$$

Dividing by $\mu(B)$, we have

$$(*) \qquad \left| \frac{\eta_1 * \eta_2(B)}{\mu(B)} - h_1(v_1)h_2(v_2) \right| \leq \epsilon(|h_1(v_1)| + |h_2(v_2)|).$$

As ϵ tends to zero, so do $\delta(\epsilon)$ and $f(\delta)$, and thus the radius of the ball does as well. So $(d(\eta_1 * \eta_2)/d\mu)(v) = h_1(v_1)h_2(v_2)$.

We require only a minor modification to deal with the sequence $\{z_j\}$. Pick $\epsilon > 0$; there exist j_0 such that $j \geq j_0$ implies $|v - z_j| \leq \delta(\epsilon)/2$. For $j > j_0$, let B be any open set containing z_j and contained in $z_j + B_{\delta(\epsilon)/2}$ (where B_a is the ball of radius a centred at the origin). Then $B \subseteq v + B_{\delta(\epsilon)}$ and we can apply the previous computation to obtain $(*)$ (since we are permitted to divide by $\mu(B)$ in this case as well, by hypothesis). In this case, the left fraction is an approximant for $(d(\eta_1 * \eta_2)/d\mu)(z_j)$, which we obtain by permitting the diameter of B to go to zero. ∎

The following is probably well known. As is usual, a measure on a topological space is **faithful** if its support is the whole space.

LEMMA 3.2: *Let $K = K_1 + K_2$ be a sum of compact convex bodies in $\mathbf{R}^d$ with probability measures $\mu = \mu_1 * \mu_2$ where* cvx supp $\mu_i = K_i$. *Let $h_i : K_i \to \mathbf{R}$ be continuous, and define signed measures via $d\eta_i = h_i \, d\mu_i$. Suppose in addition that $d\mu_i = H_i \, d\lambda$ where λ is Lebesgue measure, $H_i : K_i \to \mathbf{R}^+$ are continuous, and H_1 vanishes on the boundary of K_1. Suppose in addition that μ is a faithful measure on K. Then*

$$\frac{d(\eta_1 * \eta_2)}{d\mu}$$

is a function continuous on the interior of K.

Proof: We note that H_1 extends to a continuous function on all of $\mathbf{R}^d$ by setting it to be zero outside K_1. Thus $h_1 H_1$ also extends to a continuous function on all of $\mathbf{R}^d$. Then $d\eta_1 = h_1 H_1 \, d\lambda$, and the Radon–Nikodym derivative is continuous (on all of $\mathbf{R}^d$), bounded, with compact support; and $d\eta_2 = h_2 H_2 \, d\lambda$ and here the Radon–Nikodym derivative is bounded, with compact support, and continuous except possibly at the boundary of K_2. Writing $d(\eta_1 * \eta_2) = h \, d\lambda$, it is well known that h is continuous. Similarly, $d(\mu_1 * \mu_2) = H \, d\lambda$ where H is continuous. Faithfulness of μ_1 ensures that H vanishes nowhere on the interior of K. Then

$$\frac{d(\eta_1 * \eta_2)}{d\mu} = \frac{\frac{d(\eta_1 * \eta_2)}{d\lambda}}{\frac{d\mu}{d\lambda}} = \frac{h}{H}$$

which is continuous on the interior of K. ∎

COROLLARY 3.3: *Let μ be a probability measure on $\mathbf{R}^d$ such that*

$$K = \operatorname{cvx}\operatorname{supp}\mu$$

is a compact body. Suppose there exists a neighbourhood U in K of the boundary of K such that μ is faithful on U. Then there exists n such that for all $m \geq n$, $\mu^{(m)} = \mu * \cdots * \mu$ is a faithful measure on mK.

Proof: If $\mu^{(n)}$ is faithful on nK, then all higher convolution powers are also faithful on the corresponding multiples of K. Hence we need only show some power is faithful on the convex hull of its support; this means we can systematically replace μ by a power of itself at any point in the argument. By compactness of the boundary, there exists a positive real number c such that if k is a point in K with $\|k - v\| \leq c$ for some v in the boundary of K, then k belongs to U. By convolving μ once with itself and shifting K (say so that the origin lies in the interior of the support), we may assume ϵK and $S_\epsilon = K \backslash (1-\epsilon)K$ are contained in the support of μ for some positive ϵ. Then the support of $\mu^{(m)}$ will contain

$$\bigcup_{a=1}^{m} \left(a\epsilon K + (m-a)S_\epsilon\right),$$

and it is not difficult to verify that this will exhaust mK if $m > 2/\epsilon$. ∎

THEOREM 3.4: *Let μ be a probability measure defined on $\mathbf{R}^d$ with the following properties.*

(a) *$K := \operatorname{cvx}\operatorname{supp}\mu$ is a strictly convex compact body in $\mathbf{R}^d$;*

(b) *μ is absolutely continuous with respect to Lebesgue measure.*

Then $G_{\text{cont}}(\mu)$ exists and is a partially ordered algebra.

Proof: Since $\operatorname{cvx}\operatorname{supp}\mu^{(m)} = mK$, we may replace μ by a convolution power of itself at any point. From absolute continuity, convolving μ with itself a couple of times will allow us to assume that it has continuous Radon–Nikodym derivative with respect to Lebesgue measure. A further convolution will allow us to assume that the Radon–Nikodym derivative is strictly positive on the interior of a neighbourhood of the boundary. By Corollary 3.3, a further convolution power will be faithful, and yet another will guarantee that the current Radon–Nikodym derivative be strictly positive on the interior of the convex body. So we are in the situation that $d\mu^{(n)}/d\lambda = H$ where H is strictly positive on the interior of nK, and of course vanishes on the boundary, and is continuous (it of course extends to a continuous function on all of $\mathbf{R}^d$); moreover, these properties persist if n is increased. With H playing the role of H_1 in Lemma 3.2, we have that $f = d(\eta_1 * \eta_2)/d\mu^{(n+1)}$ is continuous on the interior of $(n+1)K$. Every boundary point of a strictly convex compact set is exposed, so by Lemma 3.1, f is continuous on all of $(n+1)K$. ∎

The strictly convex situation is perhaps disjoint from the original context of the finitely supported case. It is considerably easier to give an existence result when there is "enough" measure attached to the boundary. On the other hand, when there is not enough (or as more frequently happens, none at all) measure assigned to the boundary, the conclusion of Theorem 3.4 actually fails, even when $d\mu/d\lambda$ is C^∞ and vanishes nowhere on the interior of K (in fact for K a square), Example 6.1. In order to obtain existence theorems for more general convex bodies, a somewhat more complicated development is required, examining the behaviour near the boundary.

Let F be an exposed face of K with $\dim F = f$; we assume $f > 0$, so F is not a singleton. The set of vectors in the unit ball of $\mathbf{R}^d$ that expose F spans a subspace of dimension $d - f$ in which it is open. Let u be any unit vector that exposes F, normalized so that $u \cdot w = 1$ for w in F. For $0 < \delta$, define $A_\delta = \{ w \in K \mid u \cdot w \geq \max_{w' \in K} \{u \cdot w'\} - \delta \}$. Define probability measures μ_δ on K via

$$\int_{w \in K} h(w)\, d\mu_\delta = \frac{\int_{A_\delta} h(w)\, d\mu(w)}{\mu(A_\delta)}$$

for continuous $h: K \to \mathbf{R}$. It is completely routine to verify that any limit point (in the weak topology) of a countable subsequence of $\{\mu_\delta\}$, with the δs tending to 0 must be supported on F. In general, it is not true that $\{\mu_\delta\}$ converges to a measure even when μ is absolutely continuous with respect to Lebesgue measure and its Radon–Nikodym derivative is C^∞. Let us assume that $\{\mu_\delta\}_{\delta \to 0}$ converges to a measure μ_F. (Simple examples with $d = 3$ and $f = 1$ reveal that μ_F depends on the choice of exposing vector u and not just the face F). This is often easy to verify.

If F consists of a single vertex, then μ_F must be the point mass, and of course convergence does occur (no matter which μ we pick!). This is what makes the strictly convex situation much easier to deal with than the more general case.

Compare the following with Lemma 3.1.

LEMMA 3.5: *Let F be an exposed face of dimension exceeding zero, of the compact convex body K; let μ be a probability measure on K with* cvx supp $\mu = K$. *Suppose $\{\mu_\delta\}_{\delta \to 0} \to \mu_F$, and moreover assume that μ_F has no mass on the boundary of F. Let m and n be positive integers, and let $h_1 : mK \to \mathbf{R}$ and $h_2 : nK \to \mathbf{R}$ be continuous functions, with corresponding signed measures η_i defined by $d\eta_1 = h_1 \, d\mu^{(m)}$ and $d\eta_2 = h_2 \, d\mu^{(n)}$. Let $\eta_{i,F}$ be the signed measures given by $d\eta_{1,F} = h_1|(mF) \, d\mu_F^{(m)}$ and $d\eta_{2,F} = h_2|(nF) \, d\mu_F^{(n)}$. Then for all v in $(m+n)F$,*

$$\frac{d(\eta_1 * \eta_2)}{d\mu^{(m+n)}}(v) = \frac{d(\eta_{1,F} * \eta_{2,F})}{d\mu_F^{(m+n)}}(v).$$

Let z_j be a sequence of points of $(m+n)K$ converging to a point v in $(m+n)F$ with the property that $\mu^{(m+n)}$ does not kill any neighbourhood of any z_j. Then

$$\lim_j \frac{d(\eta_1 * \eta_2)}{d\mu^{(m+n)}}(z_j) = \frac{d(\eta_{1,F} * \eta_{2,F})}{d\mu_F^{(m+n)}}(v).$$

Proof: Let $\tilde{F}$ denote the affine span of F, so that $\tilde{F}$ is a translate of a subspace of $\mathbf{R}^d$, say having dimension f with $0 < f < d$. Fix a unit vector u exposing F with respect to K, and define $A_{\epsilon,m} = \{ w \in mK \mid u \cdot w \geq (\max_{z \in mK} u \cdot z) - \epsilon \}$. For $\epsilon > 0$ there exists $t(\epsilon) > 0$ (depending on m, but in an obvious way), such that $\operatorname{dist}(w, F) < t(\epsilon)$ for all w in $A_{\epsilon,m}$ with $t(\epsilon) \to 0$ as $\epsilon \to 0$. Define $P_\epsilon : A_{\epsilon,m} \to mF$ via $P_\epsilon(w) = w_F$ where w_F is the closest point in mF to w. Then P_ϵ (depends on m) is continuous. For continuous $h : mK \to \mathbf{R}$, define $h^\epsilon : A_{\epsilon,m} \to \mathbf{R}$ via $h^\epsilon(w) = h(w_F)$. Uniform continuity of h yields $\|h - h^\epsilon\|_{A_{\epsilon,m}}^\infty < s(\epsilon)$ where $s(\epsilon) \to 0$ as $\epsilon \to 0$; obviously s depends on h.

Now apply the preceding with $m = m(i)$. If w_i belong to $m(i)K$ respectively and $w_1 + w_2$ lies in $A_{\epsilon,m(1)+m(2)}$, then it is immediate that w_i belongs to $A_{\epsilon,m(i)}$. Hence for $W \subset A_{\epsilon,m(1)+m(2)}$,

$$\begin{aligned} \eta_1 * \eta_2(W) &= \int_K \eta_1(W - z) \, d\eta_2(z) \\ &= \int_{A_{\epsilon,m(2)}} \int_{W-z} h_1(x) h_2(z) \, d\mu^{(m(1))}(x) \, d\mu^{(m(2))}(z). \end{aligned}$$

If we let W range over a family of balls with radii converging to zero and centred at the same point, y, then we obtain

$$\frac{d(\eta_1 * \eta_2)}{d\mu^{(m(1)+m(2))}}(y) = \int_{A_{\epsilon,m(2)}} h_1(y-z) h_2(z) \, d\mu^{(m(2))}(z).$$

The absolute value of the difference between this and

$$\int_{A_{\epsilon,m(2)}} h_1^\epsilon(y-z)h_2^\epsilon(z)\,d\mu^{(m(2))}(z)$$

is bounded above by $\mu^{(m(2))}(A_{\epsilon,m(2)})(\|h_1\|s_2(\epsilon)+\|h_2\|s_1(\epsilon))$ (using the supremum norm). For y in $(m(1)+m(2))F$, we deduce that as $\epsilon \to 0$,

$$\frac{\int_{A_{\epsilon,m(2)}} h_1(y-z)h_2(z)\,d\mu^{(m(2))}(z)}{\mu^{(m(2))}(A_{\epsilon,m(2)})} \to \int_{m(2)F} h_1(y-z)h_2(z)\,d\mu_F(z).$$

This yields (with $y=v$) the first conclusion.

For the second, set $y=z_j$; as $z_j \to 0$, the corresponding ϵ (required so that $A_{\epsilon,m(2)}$ contain z_j) go to zero and uniform continuity of h_i yield the result. ∎

The argument in the proof can be modified if we merely require that $\mu_F^{(n)}(\partial(nF)) \to 0$ as $n \to \infty$, instead of insisting that the measure of the boundary of F be zero.

THEOREM 3.6: *Suppose that μ is a probability measure defined on $\mathbf{R}^d$ such that*

(a) *$K :=$ cvx supp μ is a compact convex body;*

(b) *μ is absolutely continuous with respect to Lebesgue measure;*

(c) *every face F of dimension exceeding zero of K is exposed and there is a choice of exposing vector so that μ_ϵ converges (weakly) to a measure μ_F on F such that $\mu_F(\partial F)=0$.*

Then $G_{\rm cont}(\mu)$ exists and is a partially ordered algebra.

Proof: The smoothing argument in Theorem 3.4 permits us to assume the R-N derivative of μ with respect to Lebesgue measure is continuous and vanishes on the boundary; then continuity on the interior of the relevant Radon–Nikodym derivatives follows as in that proof, and continuity on the boundary follows from Lemma 3.5. ∎

It is reasonable to conjecture that if $G_{\rm cont}(\mu)$ exists and is a partially ordered algebra, then for every exposed face F, μ_F exists.

Convergence of μ_ϵ is an interesting property. All limit points will automatically be supported on F, so the limit measure, if it exists, will be a probability measure on F. For instance, suppose $d\mu = h\,d\lambda_K$, $K=$ cvx supp h, $h\colon K \to \mathbf{R}$ is continuous and does not vanish identically on F; say the latter is of dimension f. A plausible guess is that $\{\mu_\epsilon\}$ converges, and it should converge to the measure obtained by normalizing f-dimensional Lebesgue measure on F weighted by $h|_F$. If $f = d-1$ (i.e., F is a facet, a face of codimension one), then F is automatically exposed (as are all maximal proper faces), and a simple argument will be given to show that both parts of the guess are true. However, if $f < d-1$, it is easy to give examples where μ_ϵ converges, but the limit will not have R–N derivative $h|F$; moreover, the limit measure depends on the choice of exposing vector u. (In three dimensions, take a wedge which is obtuse at the front and very sharp, i.e., acute, at the back, with the face being the obvious edge, and with $h \equiv 1$; there is more mass at the front—the cross-sections are bigger—than at the back, so any limit measure will have more mass near the front than the back, and thus cannot be Lebesgue measure.)

If h is identically zero on the face, then $\{\mu_\epsilon\}$ need not converge at all (Example 6.1). If F is a facet and h is the restriction of a function real analytic on a neighbourhood of F in $\mathbf{R}^d$, then an easy argument reveals that the limiting measure exists and is given by renormalizing $(u \cdot \nabla h)|_F$ (where u exposes F) if the latter is not identically zero; otherwise take a sufficiently high directional derivative. For lower-dimensional faces, there are problems in deciding convergence; this is already apparent in the argument of Lemma 3.5.

To obtain some results about convergence of $\{\mu_\epsilon\}$, fix a compact convex body K with measure μ satisfying $\operatorname{cvx}\operatorname{supp}\mu = K$, that is absolutely continuous with respect to Lebesgue measure; suppose h is the R–N derivative (with respect to Lebesgue measure). Let F be an f-dimensional face exposed by the unit vector u, and form A_ϵ as above, together with the flat

$$K_\epsilon = \left\{ k \in K \mid u \cdot k = (\max_{w \in K} u \cdot w) - \epsilon \right\}.$$

The boundary of A_ϵ is K_ϵ together with (for ϵ sufficiently small) the graph of a function $g_\epsilon : K_\epsilon \to \mathbf{R}^+$. The graph of g is just the portion of the boundary of K cut off by K_ϵ. Since A_ϵ is convex, $-g$ is convex (as a function), and so g is continuous on the relative interior of K_ϵ (a $d-1$-dimensional space) [F; Theorem 3.5, p. 110]. For e in F, let $C_{e,\epsilon}$ be the $d-f$-dimensional cross-section obtained by intersection A_ϵ with the affine space passing through e orthogonal to F. Possibly for some points on the boundary, $C_{e,\epsilon}$ will be a singleton; to avoid this and other pathologies, we observe that for any closed neighbourhood in the relative interior of F, for all sufficiently small $\epsilon > 0$, not only is $C_{e,\epsilon} \cap K_\epsilon$ of biggest possible dimension, $d-f-1$, but $C_{e,\epsilon}$ is the convex hull of K_ϵ with the graph of $g|_{K_\epsilon}$, for all e in the neighbourhood. Again by making ϵ small enough, we can also assume that $C_{e,\epsilon}$ consists of precisely the points w in A_ϵ for which e is the nearest point in F to w. Let B be such a closed neighbourhood; we are free to increase B so that $\lambda_F(B)$ is arbitrarily close to 1 (in order to have the other properties holding, this will require reducing ϵ).

Write $d\mu = h\,d\lambda$. Let $\tilde{\lambda}$ denote $d-f$-dimensional Lebesgue measure, and let $q: K \to \mathbf{R}$ be continuous. We wish to investigate the behaviour (as $\epsilon \to 0$ and as B tends to F) of

$$\mu_\epsilon(q) := \frac{\int_{A_\epsilon} qh\,d\lambda}{\int_{A_\epsilon} h\,d\lambda}. \tag{1}$$

By Fubini, for any continuous $p: K \to \mathbf{R}$,

$$\int_B p\,d\lambda = \int_{e\in F}\int_{C_{e,\epsilon}} p\,d\tilde{\lambda}\,d\lambda_F(e). \tag{2}$$

As p is uniformly continuous on K, there exists $s(\epsilon)$ (depending on p) such that $|p(w) - p(w_F)| < s(\epsilon)$, where the latter goes to zero as ϵ does. The expression (2) is approximated by $\int_F p(e)\,d\tilde{\lambda}(C_{e,\epsilon})$ with an error of at most $\int_F s(\epsilon)\tilde{\lambda}(C_{e,\epsilon})\,d\lambda_F(e) = s(\epsilon)\lambda(B)$. Apply this in (1) with $p = qh$ and $p = h$.

Set $M = \int_B q|F\ h|F\,d\lambda_F$, $N = \int_B h|F\,d\lambda_F$, $\delta_1 = \int_{A_\epsilon} qh\,d\lambda - \int_B q|F\ h|F\,d\lambda_F$, and $\delta_2 = \int_{A_\epsilon} h\,d\lambda - \int_B\ h|F\,d\lambda_F$. Then

$$\begin{aligned} \frac{\int_{A_\epsilon} qh\,d\lambda}{\int_{A_\epsilon} h\,d\lambda} - \frac{\int_B q|F\ h|F\,d\lambda_F}{\int_B\ h|F\,d\lambda_F} &= \frac{M+\delta_1}{N+\delta_2} - \frac{M}{N} \\ &= \frac{\delta_1}{N+\delta_2} - \frac{M}{N}\frac{\delta_2}{N+\delta_2}. \end{aligned} \tag{3}$$

We are assuming h is nonnegative, and at this point we must assume $h|F$ is not identically zero. We show that δ_1/N and δ_2/N go to zero as ϵ does. However, δ_1 is at most $s_{qh}(\epsilon)\lambda(A_\epsilon)$, δ_2 behaves similarly, and

$$N = \int_B (h|F)\,\tilde{\lambda}(C_{e,\epsilon})d\lambda_F.$$

By picking a neighbourhood in B where h is greater than some number κ, we easily obtain N exceeds some multiple of $\lambda(A_\epsilon)$, the multiple not depending on ϵ. So both δ_i/N tend to zero, and thus the difference in (3) tends to zero.

Consider the normalized distributions (i.e., nonnegative functions whose integrals are 1) on F,

$$D_{e,\epsilon} = \tilde{\lambda}(C_{e,\epsilon}) \Big/ \int_F C_{e,\epsilon}\, d\lambda_F(e).$$

For ϵ fixed, the function $D_\epsilon\colon \mathrm{Int}(F) \to \mathbf{R}^+$ is continuous. If D_ϵ converges pointwise almost everywhere (as ϵ tends to zero), say to D, then $\{\mu_\epsilon\}$ will converge to the measure on F whose R–N derivative with respect to to λ_F is D, and we will be done. (Observe that we are not required to show that D is continuous, simply that the measures given by D_ϵ converge weakly.) So it remains to investigate conditions under which this convergence occurs. (Note that we can expand the domain of D_ϵ to the relative interior of F, by continuously enlarging the closed neighbourhood B as ϵ shrinks.)

If K is a polytope (or is locally a polytope, around the face F), then in fact for all e in the relative interior of F and sufficiently small ϵ (depending on e), and all t in the interval $(0,1)$,

$$\tilde{\lambda}(C_{e,t\epsilon}) = t^{d-f}\tilde{\lambda}(C_{e,\epsilon}).$$

To see this, we pick our closed neighbourhood B in F and by selecting ϵ' sufficiently small, for all e in B, the set of points in $A_{\epsilon'}$ nearest to e lies on the orthogonal set $C_{e,\epsilon'}$, which is itself a polyhedron and a cone with base in $K_{\epsilon'}$. Dilating ϵ' just truncates the cone, so its $d-f$-dimensional volume is multiplied by the amount of the dilation to the $d-f$ power. Hence for each e in the interior of F, $\{D_{e,\epsilon}\}$ is eventually constant (as ϵ goes to zero), and this is more than enough to guarantee pointwise convergence a.e.

The upshot of this is that if K is a polytope and h does not vanish identically on a face of dimension exceeding 0, then μ_ϵ does converge to a measure on F. (If F is zero dimensional, convergence is automatic, as we have seen.) Hence:

Corollary 3.7: *Let K be a compact convex polytope with interior in $\mathbf{R}^d$ and let μ be a probability measure on K such that* cvx supp $\mu = K$ *and $d\mu = h\,d\lambda$, where $h\colon K \to \mathbf{R}$ is continuous and does not vanish identically on any proper face of dimension one or more. Then $G_{\mathrm{cont}}(\mu)$ exists and is a partially ordered algebra. Moreover, for each face F of dimension exceeding zero, μ_F exists and contains* supp $h|F$ *in its support.*

Since convergence of $\{\mu_\epsilon\}$ is linked inextricably with the cross-sectional volume function $\tilde{\lambda}(C_{e,\epsilon})$, it is worth investigating the behaviour of the latter as ϵ goes to zero, for more general convex bodies. For example, if the face is of codimension 1, then $C_{e,\epsilon}$ is simply a line segment of length proportional to ϵ, and the D_ϵ are again ultimately stationary for every e in the relative interior of F (one can think of this case and the zero dimensional case as special cases of "polyhedral" cones, because the arguments parallel those for polytopes).

V. Zurkowski observed that any limit measure of $\{D_\epsilon\}$ has a continuous R–N derivative that cannot vanish on the relative interior of F.

If $h|F$ vanishes, it can still happen that $\{\mu_\epsilon\}$ admits a unique limit measure. For example, suppose F is a facet exposed by the unit vector u, $h|F = 0$ and $(u \cdot \nabla h)|F$ does not vanish identically. Then $\{\mu_\epsilon\}$ converges to the measure on F with R-N derivative $(u \cdot \nabla h)|F$ renormalized. To see this, approximate A_ϵ by B as we did before, so that the cross-sections $C_{e,\epsilon}$ are just line segments of the same length l (depending linearly on ϵ). Integration over $P_\epsilon^{-1}(B)$ is quite easy, since it is a product space (B with an interval)—for $s: K \to \mathbf{R}$ continuous, $\int_F \int_{t=0}^{\epsilon} s\, dtd\lambda_F$ equals $\int_F \int_{t=0}^{\epsilon} s(e-ut)\, dtd\lambda_F(e)$, so $\int_{A_\epsilon} h\, d\lambda$ is sufficiently closely approximated by $\epsilon \int_F \int_{t=0}^{\epsilon} h(e-ut)/\epsilon\, dtd\lambda_F(e)$, and this closely approximates $\epsilon \int_F (u \cdot \nabla h)|F\, d\lambda_F$, etc. (the details are routine). However, when F is lower dimensional, it is not clear what happens.

4. Perfidious traces

Here we show that in many cases, the map from the pure trace space of $G \equiv G_{\mathrm{cont}}(\mu)$ to $K = \operatorname{cvx} \operatorname{supp} \mu$, $\Gamma: \mathrm{T_e}(G) \to K$, is a homeomorphism, and describe how pure traces arise as limits of point evaluation traces.

Let u and a be points in $\mathbf{R}^d$, with u nonzero. Define the ray, $X_{u,a}: [0,\infty) \to \mathbf{R}^d$ via $X(t) \equiv X_{u,a}(t) = tu + a$. We create a one parameter family of pure traces on each of $G_{\mathrm{ac}}(\mu)$, $G_\infty(\mu)$, and $G_{\mathrm{cont}}(\mu)$ (in the last case, if it exists),

$$\gamma_{X(t)}\big([\nu, k]\big) = \frac{\int_{\mathbf{R}^d} \exp(X(t) \cdot w)\, d\nu(w)}{\int_{\mathbf{R}^d} \exp(X(t) \cdot w)\, d\mu(w)}.$$

Then $\Gamma^\infty(\gamma_{X(t)})$, $\Gamma(\gamma_{X(t)})$ are paths in K. It turns out that all the limit points of these paths (as $t \to \infty$) lie in the face of K exposed by u, that is, in the face

$$\left\{ w' \in K \mid u \cdot w' = \max_{w \in K} u \cdot w \right\}.$$

We have something a bit stronger; this is almost entirely in [H4; Lemma E8, p. 125], but the proof is included here because of conflicting notation.

LEMMA 4.1: *Let $\{r_k\}$ be an sequence of points in $\mathbf{R}^d$ such that y is a limit point of $\{\Gamma^\infty(\gamma_{r_k})\}$ in K, and $\liminf \|r_k\|_2 = \infty$. Then y belongs to the union of the faces of K exposed by the limit points (in the unit sphere of $\mathbf{R}^d$) of $\left\{\frac{r_k}{\|r_k\|_2}\right\}$.*

Proof: Define probability measures on K, $d\mu_k$ by normalizing $\exp(r_k \cdot -)\,d\mu$ so that its mass is one. Choose a limit point of this family of probability measures (in the weak topology), ν. We show that the support of ν will be in the face exposed by a limit of $\{r_k/\|r_k\|_2\}$.

Form $u_k = r_k/\|r_k\|_2$; we may refine the sequence $\{r_k\}$ so that $d\mu_k$ converges to ν, and let u be a limit point of the corresponding points of the sphere, $\{r_k/\|r_k\|_2\}$; we may refine the sequence so that u_k converges to u. Let b be a real number less than $a = \sup\{\, u \cdot k \mid k \in K\}$, and let F be the face exposed by u. Let A be the closed set $\{\, w \in K \mid u \cdot w \leq b\}$; pick a real number e between a and b, and let B be the slice $\{\, w \in K \mid e \leq u \cdot w \leq a\}$. We show that $\nu(A) = 0$.

We may translate K so that $a > e > 0 > b$. By assumption, K contains interior, so $F \neq K$. Then

$$\mu_k(A) = \frac{\int_A \exp(r^k \cdot w)\,d\mu(w)}{\int_K \exp(r^k \cdot w)\,d\mu(w)} \leq \frac{\mu(A)}{\mu(B)} \frac{\sup\{\,\exp(r^k \cdot w) \mid w \in A\}}{\inf\{\,\exp(r^k \cdot w) \mid w \in B\}}. \tag{1}$$

Let $M = \sup_{w \in K} \|w\|$. Given $\epsilon > 0$, for all sufficiently large k and all w in K, $\|u_k \cdot w - u \cdot w\| < \epsilon M$. Select ϵ so that $\epsilon M < |b|$; then $u_k \cdot w < u \cdot w + \epsilon M$ for all w in A. Hence for all sufficiently large k, and all w in A, $r_k \cdot w < 0$. So the numerator is bounded by $\nu(A)$.

We also have $u_k \cdot w > u \cdot w - \epsilon M$. We may further reduce ϵ so that $\epsilon M < e/2$. Hence for w in B and all sufficiently large k, $u_k \cdot w > e/2$; for such w, $r_k \cdot w > \frac{e}{2}\|r_k\|$. So the denominator of the expression in (1) is bounded below by $\mu(B) \exp e\|r_k\|/2$; since the original set of r_k's has no bounded limit points, the refined set does not either. So as k increases, the denominator becomes arbitrarily large. Hence $\nu(A) = 0$.

Translate K back to its original position, and let b take any value less than a. We deduce that $\nu(A \backslash F) = 0$, so ν is supported on F.

Point evaluation traces are sent by Γ^∞ (and by its counterpart Γ) to the barycentres of the corresponding measures, i.e.,

$$\gamma_r \mapsto \int_K w \exp(r \cdot w)\,d\mu(w) \Big/ \int_K \exp(r \cdot w)\,d\mu(w).$$

From ν being a limit point of the corresponding measures, we have that $y = \int_K w\, d\nu$, whence y lies in F. This obviously applies for each choice u of limit point of $\left\{\frac{r_k}{\|r_k\|_2}\right\}$. ∎

Suppose γ is a pure trace of $G_{\text{cont}}(\mu)$ (which we now assume to exist). Then we obtain a compatible family of positive linear functionals, $\rho_n : C(nK) \to \mathbf{R}$ obtained from the maps $C(nK) \to C(nK, \mu^{(n)}) \to G_{\text{cont}}(\mu)$. We often use ρ_n to denote the corresponding probability measure it induces on nK. It is reasonable to call $\{\rho_n\}$ a **harmonic family of measures.**

Let u and a be a pair of vectors in $\mathbf{R}^d$ with u not zero, and suppose that u exposes a zero-dimensional face, i.e., an exposed point v of K. For each integer n define the positive linear functionals, ρ_n as the point mass at nv (the corresponding vertex of nK). We claim (a) that $\{\rho_n\}$ defines a multiplicative trace on G_{cont}, and (b) that the resulting pure trace is obtained as the limit of $\gamma_{X_{u,a}(t)}$ $(t \to \infty)$.

Without loss of generality, we may assume $\|u\|_2 = 1$. Pick a strictly increasing unbounded sequence $\{t_k\}_{k \in \mathbf{N}}$ of positive numbers, and set $r_k = t_k u + a$; then $u_k = r_k / \|r_k\|$ converges to u. Then Lemma 4.1 applies, and we deduce that the limit point y must be v. We can also apply this with μ replaced by $\mu^{(n)}$ for any positive integer n, so that v will be replaced by nv. Let ν be a signed measure on nK such that $d\nu = h\, d\mu^{(n)}$. Set $\gamma^k = \gamma_{r_k}$; then

$$\gamma^k([\nu, n]) = \frac{\int_{nK} h(w) \exp((t_k u + a) \cdot w)\, d\mu^{(n)}(w)}{\int_{nK} \exp((t_k u + a) \cdot w)\, d\mu^{(n)}(w)} = \int_{nK} h(w)\, d\mu_k(w), \tag{2}$$

where μ_k is defined as in the proof of Lemma 4.1, with respect to $\mu^{(n)}$. Any limiting point of the probability measures μ_k must have its support in the face exposed by u, in this case, the singleton consisting of nv. Hence these converge to the point mass, and so the limit, as k increases, of the terms in (2) is simply $h(nv)$. This simultaneously proves (a) and (b).

The upshot is that given an exposed point v of K, there is a unique multiplicative trace γ of G_{cont} such that both $\Gamma(\gamma) = v$ and γ is a limit of point evaluation traces, its corresponding sequence of measures consists of the point masses at nv, and it can be obtained as the limit of point evaluation traces by taking any ray in $\mathbf{R}^d$ whose directional derivative exposes v as a vertex of K.

In the case that K is strictly convex, every boundary point of K is a vertex, so we obtain a family of pure traces such that Γ induces a bijection between it and the boundary of K. We now proceed to show that Γ is a homeomorphism between the pure trace space, $\mathrm{T_e}(G_{\mathrm{cont}}(\mu))$, and K. A particular consequence is that the point evaluation traces are dense in the pure trace space (a property which is neither obvious nor generally holds, even for reasonable choices of μ). Whenever this phenomenon occurs it follows that Γ acts by sending traces to their "barycentres". Explicitly, if γ is a pure trace, it induces a positive linear functional on the first level, $C(K,\mu)$, hence a probability measure ρ on K; then $\Gamma(\gamma)$ is the barycentre of ρ, that is, $\int_K w\,d\rho$. This is not obviously a consequence of the definition of Γ.

Proposition 4.2: *Let μ be a probability measure on $\mathbf{R}^d$ such that*

$$K = \operatorname{cvx}\operatorname{supp}\mu$$

is a compact convex body. Suppose that $G_{\mathrm{cont}}(\mu)$ exists. Let γ be a pure trace that is not faithful, and let $\{\rho_n\}$ be its corresponding harmonic family of measures. There exists a proper face, F, of K such that ρ_n has its support in nF, and additionally for all sufficiently large n, $\operatorname{cvx}\operatorname{supp}\rho_n$ contains a point in the relative interior of nF.

Proof: By hypothesis, $\gamma[\eta,n] = 0$ for some n and nonzero positive measure η with $d\eta = h\,d\mu^{(n)}$ and $h{:}\,nK \to \mathbf{R}$ continuous. If W is the cozero set of h (i.e., $W = \{\,w \in K \mid h(w) > 0\,\}$), then obviously $\rho_n(W) = 0$ and for all m, $\rho_{n+m}(W + mK) = 0$. Suppose that Y is an open subset of pK (for some p) whose closure lies in the interior of pK. Then there exists an integer $M > n$ such that $MY \subseteq W + (Mp-n)K$ (the distance from MY to the boundary of MpK becomes arbitrarily large). There exists a continuous nonnegative function g with support contained in W such that if g_M is the Radon–Nikodym derivative of $\varphi{*}\mu^{(Mp-n)}$ (where $d\varphi = g\,d\mu^{(n)}$), then $MY \subseteq g_M^{-1}([1-\delta,1])$ for some δ between 0 and 1. Define η_M via $d\eta_M = g_M\,d\mu^{(Mp)}$. If π is a measure with $d\pi = f\,d\mu^{(p)}$ and the continuous function f is supported in Y, then $\pi^{(M)} \leq N\eta_M$ for some integer N. Hence $[\pi^{(M)}, Mp] \leq N[\eta_M, Mp]$. Since g is supported in a set that is killed by ρ_n, $\gamma[\eta_M, Mp] = \gamma[\varphi, n] = 0$. Thus $\gamma[\pi^{(M)}, Mp] = 0$. Since γ is multiplicative, $\gamma[\pi,p] = 0$. Hence $\rho_p(Y) = 0$.

So any open set with closure in the interior of pK is killed by ρ_p, for any p. Hence the support of ρ_p is contained in the boundary of pK. Next we show that cvx supp ρ_p is contained in the boundary, which is another way of saying supp ρ_p is contained in a proper face. Let F and F' be faces of K whose sum contains a d-ball. Let U and V be disjoint closed subsets of the two faces pF, pF' respectively, such that $U \cap pF'$ and $V \cap pF$ are empty, and we can additionally choose them so that $U + V$ itself contains a d-ball. There exist nonnegative continuous functions f, f' respectively on pK such that f is zero on V, one on U and very small off a small neighbourhood of U, f' satisfies these properties with U and V interchanged, and $\operatorname{supp} f \cap \partial pK \subset pF$, $\operatorname{supp} f' \cap \partial pK \subset pF'$. Form the corresponding measures via $d\eta_1 = f\, d\mu^{(p)}$ and $d\eta_2 = f'\, d\mu^{(p)}$, and convolve them.

Suppose that each $\gamma[\eta_i, p] \neq 0$. By multiplicativity, $\gamma[\eta_1 * \eta_2, 2p] \neq 0$. Observing that $U + V$ contains a d-ball, it is easy to see that ρ_{2p} contains an interior point in its support. This is a contradiction. Hence the support of ρ_p is contained in a proper face. For each p, there exists a minimal face F_p of K such that pF_p contains the support of ρ_p. The same addition argument yields that all the F_p must be the same for all sufficiently large p. ∎

THEOREM 4.3: *(Generalization from finitely supported measure to measures the convex hull of whose support is a strictly convex body.) Let K be a compact strictly convex body in $\mathbf{R}^d$, and let μ be a probability measure on K such that $K = $ cvx supp μ. Suppose that $G_{\text{cont}}(\mu)$ is defined (as occurs if μ is absolutely continuous) and μ is not singular. Then:*

(a) *The natural map $\Gamma: \mathrm{T_e}(G_{\text{cont}}(\mu)) \to K$ is a homeomorphism.*
(b) *The point evaluations traces are dense in the pure trace space of $G_{\text{cont}}(\mu)$.*
(c) *All non-faithful pure traces are of the form $[\eta, k] \mapsto \frac{d\eta}{d\mu^{(k)}}(kv)$ for each v in the boundary of K.*
(d) *Each of the non-faithful pure traces is obtainable as the limit (as $t \to \infty$) of point evaluations $\gamma_{X_{u,a}(t)}$, where u exposes v relative to K and a is any vector.*

Proof: (a) We have that $\Gamma: \mathrm{T_e}(G_{\text{cont}}(\mu)) \to K$ is a continuous map between compact Hausdorff spaces. It is onto, since the map restricted to the point evaluations is just the Legendre transformation, and that maps onto the interior of K. It suffices to show Γ is one to one. Pick w in K, and let γ_i be pure traces such that $\Gamma(\gamma_i) = w$.

If w lies in the interior, then by Proposition 4.2, neither γ_i can be non-faithful, so must be faithful. By Theorem 2.4, γ_i are point evaluations, say at r_i; but the value of Γ at a point evaluation at r is simply the Legendre transformation of r. Since the Legendre transformation is one to one (on points of $\mathbf{R}^d$), $r_1 = r_2$, whence γ_i are equal to each other.

If w lies in the boundary, each γ_i must be non-faithful (as extremal faithful traces are point evaluations, so are sent to a value of the Legendre transform, hence to the interior of K). Each has a corresponding family of probability measures $\rho_{n,i}$ on nK, and by Proposition 4.2, $\rho_{n,i}$ is supported in $nF(i)$ where $F(i)$ are faces of K. For a strictly convex set, the proper faces are singletons, so $F(i) = \{v_i\}$ with v_i being vertices. It is easy to check now that $\Gamma(\gamma_i)$ is thus v_i, so $v_1 = v_2$, and thus $\rho_{n,1} = \rho_{n,2}$ for all n, whence $\gamma_1 = \gamma_2$. In particular, Γ is one to one, concluding the proof that it is a homeomorphism.

Part (b) is an immediate consequence of this, Proposition 4.2, and the earlier comments, as are parts (c) and (d). ∎

If we permit line segments in the boundary of our convex bodies, more complications arise. Note that the assumption in the following that $G_{\text{cont}}(\mu)$ exist and be a partially ordered algebra is often redundant, by results of section 3 (Theorem 3.6 or Corollary 3.7).

THEOREM 4.4: *Let K be a compact convex body in $\mathbf{R}^d$, and let μ be a probability measure on K such that $G_{\text{cont}}(\mu)$ exists and is a partially ordered algebra, having the following properties:*

(a) *$K = \text{cvx supp}\,\mu$;*
(b) *μ is absolutely continuous with respect to Lebesgue measure on K and $h := d\mu/d\lambda$ is continuous as function on K;*
(c) *every face F of K is exposed and for every face of dimension exceeding 0, $\{\mu_\epsilon\}$ (see section 3) converges to a measure on F, μ_F;*
(d) *for all faces F of dimension greater than zero, $\text{cvx supp}\,\mu_F = F$;*
(e) *for all proper faces F of dimension exceeding zero, every pure trace on $G_\infty(\mu_F)$ is a point evaluation (viz. Theorem 2.4).*

Then $\Gamma: T_{\mathrm{e}}(G_{\text{cont}}(\mu)) \to K$ is a homeomorphism. In particular, the point evaluations are dense in $T_{\mathrm{e}}(G_{\text{cont}}(\mu))$.

Proof: It follows from assumptions (b) and (d) that some convolution power of μ is faithful on the corresponding multiple of K. It is convenient at various points in the argument to replace μ by a convolution power of itself; the hypotheses remain valid. As $G_{\text{cont}}(\mu)$ exists and is a partially ordered real algebra, the pure traces on it are precisely the positive multiplicative linear real-valued homomorphisms. It suffices (as in the proof of Theorem 4.3) to show Γ is one to one. By Theorem 2.4, Γ induces a bijection between the faithful pure traces and the interior points of K. So suppose γ is a pure trace that is not faithful, and let $\{\rho_n\}$ be the corresponding sequence of probability measures on $\{nK\}$ induced by γ. By Proposition 4.2, there exists a proper face F such that cvx supp ρ_n is contained in nF and contains a relative interior point thereof.

Define the following subset of $G_{\text{cont}}(\mu)$,

$$I_F = \left\{ [\nu, m] \in C(mK, \mu^{(m)}) \mid \left(\frac{d\nu}{d\mu^{(m)}}\right)\Big|_{mF} \equiv 0 \right\}.$$

It is easy to check that I_F is well-defined and a directed, convex subspace of $G_{\text{cont}}(\mu)$; it is thus an "order ideal" in it. It is also an ideal in the algebraic sense, and we can factor out I_F and impose the quotient ordering on $G_{\text{cont}}(\mu)/I_F$ so that the latter becomes a partially ordered algebra. Note that I_F is contained in the kernel of γ, so the latter induces a multiplicative, hence pure, trace $\overline{\gamma}$, on the quotient $G_{\text{cont}}(\mu)/I_F$. The idea will be that the quotient is naturally isomorphic with $G_{\text{cont}}(\mu_F)$ and $\overline{\gamma}$ will be a faithful trace on the latter.

The restriction map $C(mK, \mathbf{R}) \to C(mF, \mathbf{R})$, $g \mapsto g|F$, is compatible with the convolution operation that yields the map

$$C(mK, \mu^{(m)}) \to C((m+m')K, \mu^{(m+m')}),$$

by Lemma 3.5, with $\eta_2 = \mu^{(m')}$. It follows that $G_{\text{cont}}(\mu_F)$ exists and is a partially ordered algebra, and the map $G_{\text{cont}}(\mu) \to G_{\text{cont}}(\mu_F)$ (induced by restriction) is multiplicative; moreover, it is positive (nonnegative measures are sent to nonnegative measures). If $[\nu, m]$ in $G_{\text{cont}}(\mu)$ is in the kernel of the map, then it follows immediately that the support of ν must be disjoint from the support of $\mu_F^{(m)}$.

If supp μ_F contains a relatively open neighbourhood of ∂F, then on replacing μ and μ_F by a sufficiently high power, we can assume that supp μ_F is all of F (Corollary 3.3). Then the kernel of $G_{\text{cont}}(\mu) \to G_{\text{cont}}(\mu_F)$ consists exactly of the equivalence classes $[\nu, k]$ where $d\nu/d\mu^{(k)}$ vanishes on kF, i.e., the kernel is exactly I_F.

However, all we can assume is that the vertices of F belong to the support of μ_F. Let $\mathrm{d_e}F$ denote the set of vertices of F; $m\mathrm{d_e}F$ will denote the set of all sums of m elements of $\mathrm{d_e}F$. It is easy to see that the union,

$$\bigcup_{m \in \mathbf{N}} \frac{(m\mathrm{d_e}F)}{m}$$

is dense in F (the notation is unfortunate; the m in the numerator refers to all possible sums of m elements, while that in the denominator is supposed to divide every element of the set in the numerator by m, to bring the elements back to F). If $[\nu, k]$ is in the kernel of the map $G_{\mathrm{cont}}(\mu) \to G_{\mathrm{cont}}(\mu_F)$ with continuous $d\nu/d\mu^{(k)} = g{:}\,kK \to \mathbf{R}$, then each of $g_l := d(\nu * \mu^{(l)})/d\mu^{(k+l)}{:}\,(k+l)K \to \mathbf{R}$ is also in the kernel, and so must vanish on the support of $\mu^{(k+l)}$. For convenience, we may assume $k = 1$. Suppose $g|F$ is not zero. The support of $g|F$ contains the closure of an open ball, hence a closed convex body, call it C, so that if S_l is the set of sums of l elements of $\mathrm{d_e}F$, then $S_l + C$ (the set of all sums $s + c$ where $s \in S_l$ and $c \in C$) is contained in the support of $g_{l+1}|(l+1)F$. Now $(S_l + C)/(l+1)$ is a compact subset of F which is the closure of its interior. It is routine to show that for all sufficiently large l, there exists $\delta > 0$ such that $(S_l + C)/(l+1)$ contains a d-ball of radius δ. It follows from the density of the displayed set that for some l, $(l+1)F \cap \operatorname{supp} g_{l+1}$ is not empty. However, this means $[\nu * \mu^{(l)}, l+1]$ does not go to zero under the map to $G_{\mathrm{cont}}(\mu_F)$, a contradiction.

Hence in general, the kernel of the map $G_{\mathrm{cont}}(\mu) \to G_{\mathrm{cont}}(\mu_F)$ is contained in I_F, and the reverse inclusion is trivial (and unnecessary). Thus γ induces a positive multiplicative linear map $\overline{\gamma}{:}\,G_{\mathrm{cont}}(\mu_F) \to \mathbf{R}$. The latter is thus a pure trace.

By Proposition 4.2, if $\overline{\gamma}$ were not faithful (as a trace on $G_{\mathrm{cont}}(\mu_F)$), then its corresponding compatible family of measures would be supported on a proper subface of F. However, F was defined as the smallest face containing the support of the measure corresponding to γ, and it is easy to see that the measures for γ are the same as those for $\overline{\gamma}$. Hence the latter cannot be supported on a proper subface. Thus $\overline{\gamma}$ is faithful. By hypothesis (e) (which would be unnecessary if μ_F were known to be not singular with respect to Lebesgue measure of the corresponding dimension), $\overline{\gamma}$ must be point evaluation at some point r in the affine space of dimension that of F.

Next, we have to verify that the map $G_{\mathrm{cont}}(\mu) \to G_{\mathrm{cont}}(\mu_F)$ is compatible with the corresponding Γ and Γ_F. Let μ^i be the signed measure on K with derivative w_i (i.e., projection onto the ith coordinate), so that $\Gamma(\gamma) = (\gamma([\mu^i, 1]))$, and

$\Gamma_F(\overline{\gamma}) = (\overline{\gamma}([\mu_F^i, 1]))$. The map $G_{\mathrm{cont}}(\mu) \to G_{\mathrm{cont}}(\mu_F)$ sends $[\mu^i, 1]$ to $[\mu_F^i, 1]$, so that $\Gamma(\gamma) = \Gamma_F(\overline{\gamma})$. It follows immediately that $\Gamma(\gamma)$ lies in F. As $\overline{\gamma}$ is a point evaluation, $\Gamma_F(\overline{\gamma})$ lies in the relative interior of F, so of course, so does $\Gamma(\gamma)$.

Let γ_1 and γ_2 be two pure traces with the same image under Γ. If either is faithful, its image is in the interior, so the other one must be faithful, and thus both are given by a point evaluation, and by one to oneness of the Legendre transformation on $\mathbf{R}^d$, the point is the same, so the traces are the same. Hence both must be unfaithful with image in the boundary of K. If we define F_1 and F_2 as we did F above (for the generic γ), by the previous paragraph, $\Gamma(\gamma_j)$ lies in the relative interior of both F_j for $j = 1, 2$; hence $F_1 = F_2$, and moreover, both $\overline{\gamma}_i$ (now known to be defined on the same quotient) must be faithful. Hence they are point evaluations arising from the same, lower dimensional affine space, and since the Legendre transformation is one to one on point evaluations, they must be point evaluations at the same point, i.e., they are the same trace. ∎

A two-dimensional example satisfying (a) through (d) but not (e), for which Γ is also not one to one is given in Example 6.3. In this case, the measure μ is a little strange, but absolutely continuous.

In analogy with Theorem 4.3, it should be true that the image of the path of point evaluations $\left\{\Gamma(\gamma_{X_{u,a}(t)})\right\}_{t\to\infty}$ converges to $\int_F w \exp(a \cdot w)\, d\mu_F(w)$.

COROLLARY 4.5: *Let K be a compact convex polytope with interior in $\mathbf{R}^d$, and let μ be a probability measure on K such that cvx $\operatorname{supp}\mu = K$ and $h := d\mu/d\lambda$ is a continuous function on K such that for each face F of dimension exceeding zero, cvx $\operatorname{supp}(h|F) = F$. Then $\Gamma: T_{\mathrm{e}}(G_{\mathrm{cont}}(\mu)) \to K$ is a homeomorphism.*

Proof: Simply apply Corollary 3.7 and the results above. ∎

5. An ergodic/density theorem

In many cases, either $G_{\mathrm{cont}}(\mu)$ is undefined or the map $\Gamma: \mathrm{T_e}(G_{\mathrm{cont}}(\mu)) \to K$ is not a homeomorphism. When it is a homeomorphism, the point evaluations are dense in the space-time boundary, $\mathrm{T_e}(G_{\mathrm{cont}}(\mu))$. We can ask whether the point evaluation traces are dense in $\mathrm{T_e}(G_\infty(\mu))$, which is always defined. A formulation that characterizes density is derivable from dimension group techniques; however, the property appears to be more interesting as a consequence of density, rather than as a precursor. Its form is an equality of two norms, one concerning space-time and the other spatial, and resembles the ergodic theorem.

THEOREM 5.1: *Let μ be a probability measure on $\mathbf{R}^d$ with compact support, and set $K = \mathrm{cvx}\,\mathrm{supp}\,\mu$; suppose K contains interior. Let G be one of $G_{\mathrm{cont}}(\mu)$ (if it exists) or $G_\infty(\mu)$. Then the set of point evaluations is dense in the pure trace space $T_e(G)$ if and only for every nonnegative measure ν on mK such that $[\nu, m]$ belongs to G,*

$$(1) \qquad \lim_{n\to\infty} \sup_{w\in(m+n)K} \frac{d(\nu * \mu^{(n)})}{d\mu^{(m+n)}}(w) = \sup_{r\in\mathbf{R}^d} \frac{\int_{mK} \exp(r\cdot w)\, d\nu(w)}{\int_{mK} \exp(r\cdot w)\, d\mu^{(m)}(w)}.$$

The left side of the expression is a limit of a sequence of L^∞ norms, taken over the spaces $(m+n)K$ as n increases; it is a general phenomenon that the sequence of norms is nonincreasing, so the limit exists in any case. The sup norm on the right is simply the supremum of $\gamma_r([\nu, m])$, the values of the point evaluations. The right side is clearly spatial, and the left side is space-time. At the moment, the only useful direction occurs when Γ is defined and a homeomorphism (so the equality occurs as a consequence of density), because I know of no non-trivial examples where the point evaluations are dense in the pure trace space of $G_\infty(\mu)$. We shall discuss what has to be proved when K is the unit interval and μ is Lebesgue measure (!) for density to occur, later.

Proof of Theorem 5.1: As we noticed in the preceding paragraph, the right side is $\sup \gamma_r([\nu, m])$, the supremum being taken over all point evaluations. The trace space of the partially ordered unital algebra G, $\mathcal{T}(G)$, is a compact convex set, with the multiplicative traces as extreme points, so the extremal boundary, $\mathrm{T_e}(G)$, is compact. Let u denote the element $[\mu, 1]$ of G (this is the multiplicative identity element and also an order unit for G, the latter by construction). By [GH; Lemma 4.1], for an element, b of the positive cone G^+,

$$(2) \qquad \begin{aligned} \inf\left\{ \alpha\in\mathbf{R}^+ \mid b\le \alpha u\right\} &= \sup_{\gamma\in\mathcal{T}(G)} \gamma(b) \\ &= \sup_{\gamma\in\mathrm{T_e}(G)} \gamma(b) \end{aligned}$$

(we have used the vector space structure of G; this simplifies the form given in [op. cit.]). From the definition of the limit ordering, for $b = [\nu, m]$, $b \le \alpha u$ if and only if there exists n such that $\nu * \mu^{(n)} \le \alpha\mu^{(m+n)}$, or what amounts to the same thing, for all w in $(m+n)K$,

$$\frac{d(\nu * \mu^{(n)})}{d\mu^{(m+n)}(w)} \le \alpha.$$

Thus the left side of (2) is the left side of (1) with the limit replaced by lim inf. However, $\sup_w(d(\nu * \mu^{(n)})/d\mu^{(m+n)})(w)$ is non-increasing in n as a simple argument involving direct limits shows. We conclude that the left side of (1) equals $\sup_{\gamma \in \mathrm{T_e}(G)} \gamma([\nu, m])$.

If the set of point evaluations is dense in $\mathrm{T_e}(G)$, then

$$\sup_{\gamma \in \mathrm{T_e}(G)} \gamma([\nu, m]) = \sup_{r \in \mathbf{R}^d} \gamma_r([\nu, m]),$$

which is the right side of (1); so the left side equals the right side in this case.

If the set of point evaluations is not dense, there exists a positive element b of G and a trace γ such that that $\gamma(b) = 1$ but $\gamma_r(b) < 1/2$ for all point evaluations. To see this, just note that any compact subset of $\mathcal{T}(G)$, in this case, the closure of the set of point evaluations, can be separated from an extreme point by an affine continuous function [AE]. The image of G in its representation as affine continuous functions on the trace space is norm dense, so we may find b with $\gamma(b) = 1$ (it can be scaled by a real number close to 1) with $\gamma_r(b) < 1/2$ for all r in $\mathbf{R}^d$, and $1/4 < \tau(b)$ for all pure traces τ. Since G is a direct limit of unperforated groups, it is unperforated, and it follows from [EHS; Theorem 1.4], that b is positive. Hence $b = [\nu, m]$ for some nonnegative measure and some integer m. Clearly, (1) does not hold for this choice of $[\nu, m]$. ∎

Another consequence of density, is the following fairly weak statement about the positive cone in $G_{\text{cont}}(\mu)$.

PROPOSITION 5.2: *Suppose that G is one of G_∞ or $G_{\text{cont}}(\mu)$ if the latter exists, and that the point evaluations are dense in $\mathrm{T_e}(G)$. Let ν be a signed measure such that $[\nu, k]$ belongs to G and there exists $\delta > 0$ such that for all r in $\mathbf{R}^d$,*

$$\frac{\int_{\mathbf{R}^d} \exp(r \cdot w)\, d\nu(w)}{\left(\int_{\mathbf{R}^d} \exp(r \cdot w)\, d\mu(w)\right)^k} \geq \delta.$$

Then there exists n such that $\nu * \mu^{(n)}$ is a positive measure.

Proof: Density of the trace space together with the condition hypothesized guarantees that $[\nu, k]$ is strictly positive at every pure trace; as G is an unperforated partially ordered abelian group, $[\nu, k]$ is a positive element of G [EHS; 1.4], which is exactly the desired conclusion. ∎

Nothing seems to be easy about the extremely large algebras $G_\infty(\mu)$. Let μ be Lebesgue measure on the unit interval, and form $G = G_\infty(\mu)$. To prove density of the point evaluations in $\mathrm{T_e}(G)$, we would have to establish the following. Suppose ν is a signed measure on the interval $[0,m]$ and $d\nu/d\mu^{(m)}$ is essentially bounded. Suppose that

$$\frac{\int_0^m \exp(rt)\,d\nu(t)}{\left(\int_0^1 \exp(rt)\,dt\right)^m} > \epsilon \tag{3}$$

for all r in $\mathbf{R}$. Then there would have to be an integer n such that $\nu * \mu^{(n)}$ is nonnegative. One can try to find a counter-example, by letting $m = 1$ and choosing a strictly decreasing null sequence of nonnegative numbers $\{a_i\}_{i\in\mathbf{N}}$. Form $g = \sum_i (-1)^i \chi_{(a_{i+1},a_i)}$ (an alternating sum of step functions) and define ν via $d\nu = g\,d\mu$. The sequence $\{a_i\}$ can be adjusted so that (3) holds for all real r, but then it is difficult to arrange that $h * t^{n-1}$ not be nonnegative for sufficiently small t. (The problem really boils down to behaviour near the boundary.) A related attempt would involve a function such as $h(t) = \epsilon + \sin 1/t$, which has similar oscillatory properties.

6. Examples

Example 6.1: A measure μ on the rectangle $K = [0,1] \times [0,2]$ such that

(i) $\operatorname{cvx}\operatorname{supp}\mu = K$,

(ii) the Radon–Nikodym derivative of μ (with respect to Lebesgue measure, λ) is C^∞, and

(iii) $G_{\mathrm{cont}}(\mu)$ does not exist.

There exist C^∞ functions g and e defined on the unit square with the following properties:

(a) Both e and g vanish nowhere on the interior of the unit square, and their integrals over the latter are 1;

(b) all partial derivatives (all orders) of both functions tend to zero near the boundary;

(c) define $e_t, g_t\colon[0,1] \to \mathbf{R}$ via $e_t(y) = e(t,y)$ and $g_t(y) = g(t,y)$; then, as $t \to 0$,

$$\frac{\int_0^t e_t(y)\,dy}{\int_0^1 g_t(y)\,dy} \to \infty;$$

(d) for any (measurable) subset Z of $[0,1] \times [0,1/2]$,

$$\int_Z g(x,y)\,d\lambda < \int_{Z+(0,\frac{1}{2})} g(x,y)\,d\lambda.$$

Let K_1 denote the unit square, and K_2 its translate obtained by pushing it up one unit (so $K = K_1 \cup K_2$); define measure μ_i on K_i via $d\mu_1 = g\,d\lambda$ and $d\mu_2 = e(x, y-1)\,d\lambda$. From property (b), the function $f: K \to \mathbf{R}$ defined via

$$f(x,y) = \begin{cases} g(x,y) & \text{if } (x,y) \in K_1 \\ e(x,y-1) & \text{if } (x,y) \in K_2 \end{cases}$$

is C^∞ and the its support is K, as follows from (a). Now $\mu := \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ satisfies $d\mu = f\,d\lambda_K = \frac{1}{2} f\,d\lambda$. Condition (c) implies that μ becomes increasingly top heavy (more mass in a small section of K_2 than in K_1)—explicitly, for small t varying slightly, only very small subsets of $\{ (t, 1+y) \mid y \in [0,1]\}$ are required to outweigh all of $\{ (t,y) \mid y \in [0,1]\}$.

We are only interested in the behaviour of $\mu^{(n)}$ on a small piece of its support, namely K itself. On K_1, its restriction is simply the restriction of $2^{-n}\mu_1^{(n)}$, and on K_2, it is the restriction of $2^{-n}(n\mu_1^{(n-1)} * \mu_2 + \mu_1^{(n)})$. As we shall only be interested in what happens near the y-axis, the $\mu_1^{(n)}$ term in the latter expression can usually be ignored (by (c), its effect is increasingly small, as $x \to 0$, when compared with that of $\mu_1^{(n-1)} * \mu_2$ on K_2).

Consider the two line segments in K ending at $(0,1)$:

$$S(t) = (t, t+1) \qquad T(t) = (t,1) \qquad t \to 0,\ t \le 1.$$

Obviously $S(t)$ lies in K_2 and T describes the boundary between K_1 and K_2. We will establish the existence of a continuous function $h: K \to \mathbf{R}$ and corresponding measure ν on K ($d\nu = h\,d\mu$) such that for all sufficiently large n, *either*

(I) $\lim_{t\to 0} \frac{d(\nu * \mu^{(n)})}{d\mu^{(n+1)}}(S(t))$ or $\lim_{t\to 0} \frac{d(\nu * \mu^{(n)})}{d\mu^{(n+1)}}(T(t))$ does not exist, *or*

(II) the limits exist and are not equal.

Let V_t denote the rectangle in K_2 with vertices, $(0,1)$, $(t,1)$, $(0,1+t)$, and $(t, 1+t)$. For $z = (a,b)$ in the interior of K, $(z-K) \cap K = (z - nK) \cap nK$ and this is contained in the rectangle with vertices $(0,0)$, $(a,0)$, $(0,b)$, and (a,b). Let H be the Radon–Nikodym derivative of $\mu^{(n)}$ with respect to Lebesgue measure. Then

$$\begin{aligned} \frac{1}{2^n}(h * H)(S(t)) &= \int_{K_1 \cup V_t} h((t,t+1) - (x,y))\, d(n\mu_1^{(n-1)} * \mu_2 + \mu_1^{(n)}) \\ &= \int_{K_1 \cup V_t} h((t,t+1) - (x,y))\, d\mu_1^{(n)}) \\ &\quad + \int_{V_t} h((t,t+1) - (x,y))\, d(n\mu_1^{(n-1)} * \mu_2). \end{aligned}$$

Now assume that $-1 \leq h \leq 1$, h is 1 on $V_{1/4} - (0,1)$, and h is -1 off a small neighbourhood of it. For $t < 1/4$, the right integral is simply $n\mu_1^{(n-1)} * \mu_2(V_t)$; the first integral (second line of displayed equation) is over a set with proportionately very small mass—as t tends to 0, the ratio $\mu_1^{(n)}(K_1 \cup V_t)/(n\mu_1^{(n-1)} * \mu_2(V_t))$ becomes arbitrarily small. The outcome is that $2^{-n}(h * H)(S(t))$ behaves as $n\mu_1^{(n-1)} * \mu_2(V_t)$ for small t. On the other hand, $2^{-n}\mu^{(n+1)}(K \cap V_t)$ just decomposes as $n\mu_1^{(n-1)} * \mu_2(V_t) + \mu_1^{(n)}(K_1 \cap V_t)$, and the second term is small in ratio to the first. So we have

$$\lim_{t \to 0} \frac{d(\nu * \mu^{(n)})}{d\mu^{(n+1)}}(S(t)) = 1.$$

On the other hand, the limit along $T(t)$ is even easier to compute, since $((t,1) - K) \cap K$ is just the rectangle $W_t = [0,t] \times K_1$. Thus

$$\frac{1}{2^n}(h * H)(T(t)) = \int_{W_t} h((t, t+1) - (x,y))\, d\mu_1^{(n)}).$$

Property (d) implies something about μ_1—that mass moves away from the boundary on repeated convolution. Property (d) revised to

$$\mu_1^{(n)}(Z) \leq \mu_1^{(n)}(Z + (0, 1/2))$$

holds for sufficiently large n. As h is -1 over most of K_1, we deduce that $h * H(T(t))$ is eventually negative. So h has the desired properties. It follows immediately that $G_{\text{cont}}(\mu)$ does not exist. ∎

Now we give a general result in the planar case, even allowing faces that are not exposed.

THEOREM 6.2: *Let K be a planar compact convex set with interior. Let $h: K \to \mathbf{R}$ be a nonnegative continuous function such that* cvx supp $h = K$, *and such that if F is an edge (i.e., a one-dimensional face) of K for which $h|F$ is identically zero, then the directional derivative $(u \cdot \nabla h)|F$ exists and is not identically zero, where u is a vector exposing F. Then $G_{\text{cont}}(\mu)$ exists. If additionally, for every edge F,* either cvx supp $h|F = F$ *or* $h|F \equiv 0$ *but* cvx supp $((u \cdot \nabla h)|F) = F$, *then $\Gamma: T_e(G_{\text{cont}}(\mu)) \to K$ is a homeomorphism.*

Proof: We observe that all one dimensional faces are facets here and thus are exposed; in view of the discussion between Lemma 3.5 and Theorem 3.6, the only problem will be continuity of the Radon–Nikodym derivatives at an extreme but not exposed point. To this end, we adapt the argument in the case of an exposed point, namely Lemma 3.1. In place of the U_ϵ (later called A_ϵ) determined by a single exposing vector, u, we obtain a decreasing sequence of open sets determined by a suitable family of vectors which "almost exposes" the point.

Let v be an extreme but not exposed point of K. By a suitable affine linear transformation, we may assume there is a neighbourhood, V, of v in K that is the region under the graph of an increasing function $f:[-1,1] \to [0,1]$ with the following properties:

- f is concave (as an ironic twist to the definition of convex function versus convex set, it is concavity of f that guarantees convexity of the region);
- f is strictly increasing on $[-1,0]$, and $f(t) = f(0) = 1$ for $t > 0$;
- v is the point $(0,1) = (0, f(0))$ and the derivative from the left of f exists (since f is increasing) and is zero at 0 (this is precisely what makes v not exposed).

Being concave on a neighbourhood of zero, f is continuous there. Let $u_i = (a_i, b_i)$ be outward normal vectors (to be specific, "normal" means with respect to the derivatives from the left, which exist almost everywhere) to the points $v_i = (x_i, f(x_i))$, with $u_i \cdot v_i = 1$ where x_i is monotone increasing up to zero. We claim there exists a null sequence δ_i of positive numbers such that

$$\{v\} = \left\{ k \in K \mid \frac{1 - u_i \cdot k}{\delta_i} \text{ is bounded} \right\}.$$

It is immediate that whatever δ_i are chosen (going to zero), the set on the right is contained in the edge containing v; in particular, we do not have to worry about points in K outside V. We observe that u_i converges to $(0,1)$ and $a_i < 0$, $b_i > 0$. Select a point $w = (\epsilon, 1)$ on the edge, with $\epsilon > 0$. Then

$$\frac{1 - u_i \cdot w}{1 - u_i \cdot v} = \frac{1 - \epsilon a_i - b_i}{1 - b_i} = 1 + \frac{\epsilon |a_i|}{1 - b_i};$$

so it would be sufficient to show $(1 - b_i)/(-a_i) \to 0$. The "tangent" at $(x_i, f(x_i))$ has slope $-a_i/b_i$. A simple convexity argument (or just draw the diagram) yields

$$-\frac{a_i}{b_i} > \frac{1 - f(x_i)}{-x_i} > 0.$$

From $a_i x_i + b_i f(x_i) = 1$, we deduce

$$\begin{aligned}\frac{1-b_i}{-a_i} &= \frac{1-b_i f(x_i)}{-x_i} + \frac{b_i f(x_i) - b_i}{-x_i} \\ &\leq \max\left\{\frac{1-b_i f(x_i)}{-x_i}, b_i \frac{1-f(x_i)}{-x_i}\right\} \\ &\leq \max\{-a_i, -a_i\} = -a_i.\end{aligned}$$

As $a_i \to 0$, we can set $\delta_i = 1 - b_i$ and we are done.

Now we are in position to prove the analogue of Lemma 3.1 for an extreme but not exposed point v. If z_{i_k} is a convergent subsequence of $\{z_i\}$, and x_{i_k} belongs to $(z_{i_k} - K_2) \cap K_1$, then any limit point of $\{x_{i_k}\}$ can only be v; this follows from the almost exposure property just obtained. It follows immediately that there exists $\epsilon > 0$ such that for all sufficiently large i, for all x in $z_i - K_2$, $\operatorname{dist}(x - z_0) < \epsilon$. Now the argument of Lemma 3.1 can be adapted directly.

This yields that $G_{\text{cont}}(\mu)$ exists. The rest of the statement can be deduced from results in section 4. ∎

Example 6.3: A planar compact convex set K with an absolutely continuous measure μ such that $G_{\text{cont}}(\mu)$ exists, Γ is one to one on the interior, but not on the boundary; it also demonstrates the sensitivity to small changes of properties of Γ.

Let K be the convex hull of the unit disk (centred at the origin) and two translated copies, say by $(2,0)$ and $(\alpha,0)$ where $\alpha > 4$. Begin by assigning to each disk its usual Lebesgue measure. The first objection is that the Radon-Nikodym derivative of the sum (with respect to Lebesgue measure on K) will not be continuous; however all of its higher convolution powers will have absolutely continuous R–N derivatives. The next objection is that there is no mass on portions of the interior; remedy this by adding a bounded C^∞ function that is strictly positive on the interior of K and all of whose derivatives vanish on the boundary of K.

(See Illustration 6.3; K resembles a European hockey* rink;** the lightly greyed areas have the only the mass from the smooth function, while the darkly

* *Ice* hockey, of course!

** North American hockey rinks are considerably more flattened at the ends; European hockey rinks are only somewhat flattened, which in fact would make this example a little easier.

greyed areas have additionally Lebesgue measure.)

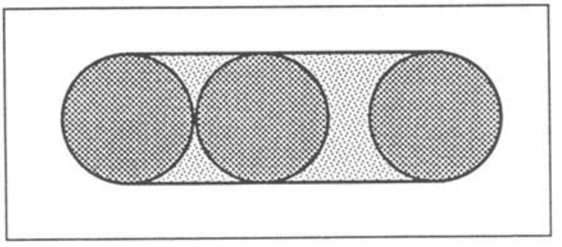

Illustration 6.3

Let F denote the top edge; similar remarks apply to the bottom edge. It is easy to check that the limiting measure μ_F exists and is just the discrete measure with equal weights at $(0,2)$, $(2,2)$, and $(\alpha,2)$. In order to use Lemma 3.5, we would require $\mu_F(\partial F)=0$ which is clearly not the case. However, the proof of the latter can be modified to accommodate the situation that $\mu_F^{(n)}(\partial(nF))\to 0$, which does hold here. In order to show $G_{\mathrm{cont}}(\mu)$ exists, it remains to deal with the four extreme but not exposed points of K. There are two remedies: either flatten the ends slightly, so all extreme points are exposed (and the resemblance to a hockey rink is increased) or deal with the extreme points as in the proof of Proposition 5.2. In either case, 3.1–3.3 will finish the argument that G_{cont} exists. (It should be true that if $G_{\mathrm{cont}}(\mu_i)$ exists for μ_1 and μ_2, then $G_{\mathrm{cont}}(\mu_1*\mu_2)$ exists as well; this would considerably simplify the example and the argument.)

To deal with Γ, we attempt to apply Theorem 4.4. Only property (e) is in doubt; if α is rational, then $G_{\mathrm{cont}}(\mu_F)$ is an algebra of certain rational functions in one variable, and it is very easy to see that (e) applies directly. However, if α is irrational, then $G_{\mathrm{cont}}(\mu_F)$ is just a special case of Example 2.5, and (e) does not apply. In fact, in the course of the proof of Theorem 4.4, it is shown (without using (e)) that $G_{\mathrm{cont}}(\mu_F)$ is a quotient of $G_{\mathrm{cont}}(\mu)$ by an order ideal, any pure trace on the quotient lifts to a pure trace on $G_{\mathrm{cont}}(\mu)$, and moreover, the two Γs are compatible. Since Γ_F is not one to one on the interior of F, it follows that Γ itself is not one to one. In particular, Γ is one to one if and only if α is rational, which demonstrates the sensitivity to small perturbations.

When α is irrational, in fact each point of the relative interior of F corresponds to a line of traces on $G_{\mathrm{cont}}(\mu)$; the only way to draw this would be to have fins extending out from each of the two edges, creating the Edsel of irrational hockey rinks!

Obviously this example could be modified by fixing the distance between the centres of the first and third disks, and letting α be the distance from the first to the second; then K remains the same for all values of the parameter, so we obtain a one parameter family of measures on a fixed compact convex set with the property that Γ is a homeomorphism if and only if the the parameter is rational. ∎

References

[AE] L. Asimow and A. J. Ellis, *Convexity Theory and its Applications in Functional Analysis*, Academic Press, New York, 1980.

[A] M. Atiyah, *Angular momentum, convex polyhedra, and algebraic geometry*, Proc. Edin. Math. Soc. **26** (1983), 121–138.

[DSW] J. L. Doob, J. L. Snell and R. E. Williamson, *Applications of boundary theory to sums of independent random variables*, in *Contributions to Probability and Statistics*, Stanford University Press, 1960, pp. 182–197.

[EHS] E. G. Effros, D. E. Handelman and C.-L. Shen, *Dimension groups and their affine representations*, Amer. J. Math. **102** (1980), 385–407.

[F] W. Fleming, *Functions of Several Variables*, second edition, Springer, Berlin, 1977.

[GH] K. R. Goodearl and D. E. Handelman, *Rank functions and K_0 of regular rings*, J. Pure Appl. Algebra **7** (1976), 195–216.

[GS] V. Guillemin and S. Sternberg, *Convexity properties of the moment mapping*, Invent. Math. **67** (1982), 491–513.

[Gu] Y. Guivarc'h, *Théorèmes quotients pour les marches aléatoires* Conference on Random Walks, Astérisque **74** (1980), 15–28.

[H1] David Handelman, *Positive polynomials and product type actions of compact groups*, Memoirs Amer. Math. Soc. **320** (1985), 1–79.

[H2] David Handelman, *Caractères virturels de groupes de Lie compacts connexes qui divisent un caractère*, C. R. Acad. Sci. Paris **302** (1986), 459–462.

[H3] David Handelman, *Deciding eventual positivity of polynomials*, Ergodic Theory & Dynamical Systems **6** (1986), 57–79.

[H4] David Handelman, *Positive polynomials, convex integral polytopes, and a random walk problem*, Lecture Notes in Mathematics **1282**, Springer-Verlag, Berlin, 1987.

[H5] David Handelman, *Extending traces on fixed point C^*-algebras under Xerox product type actions of compact Lie groups*, J. Funct. Anal. **72** (1987), 44–57.

[H6] David Handelman, *Representing polynomials by positive linear functions on compact convex polyhedra*, Pacific J. Math. **132** (1988), 35–62.

[H7] David Handelman, *Iterated multiplication of characters on generalized characters of compact connected Lie groups*, preprint (1989).

[H8] David Handelman, *Effectiveness of an affine invariant for indecomposable integral polytopes*, J. Pure Appl. Algebra **66** (1990), 165–184.

[H9] David Handelman, *Representation rings as invariants for compact groups and limit ratio theorems for them*, Int. J. Math. **4** (1993), 59–88.

[NS] P. Ney and F. Spitzer, *The Martin boundary for random walk*, Trans. Amer. Math. Soc. **121** (1966), 116–132.

[Od] T. Oda, *Convex Bodies and Algebraic Geometry*, Ergebnisse der Math., Springer-Verlag, Berlin, 1985.

[Ru] W. Rudin, *Real and Complex Analysis*, second edition, McGraw-Hill, New York, 1974.

[Ro] R. T. Rockafeller, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[Rh] O. S. Rothaus, *Some properties of Laplace transforms of measures*, Trans. Amer. Math. Soc. **131** (1968), 163–169.

[Va] F. A. Valentine, *Convex Sets*, McGraw-Hill, New York, 1964.

[VK] A. M. Vershik and S. V. Kerov, *The K-functor (Grothendieck group) of the infinite symmetric group*, J. Soviet Math. **28** (1985), 549–567.